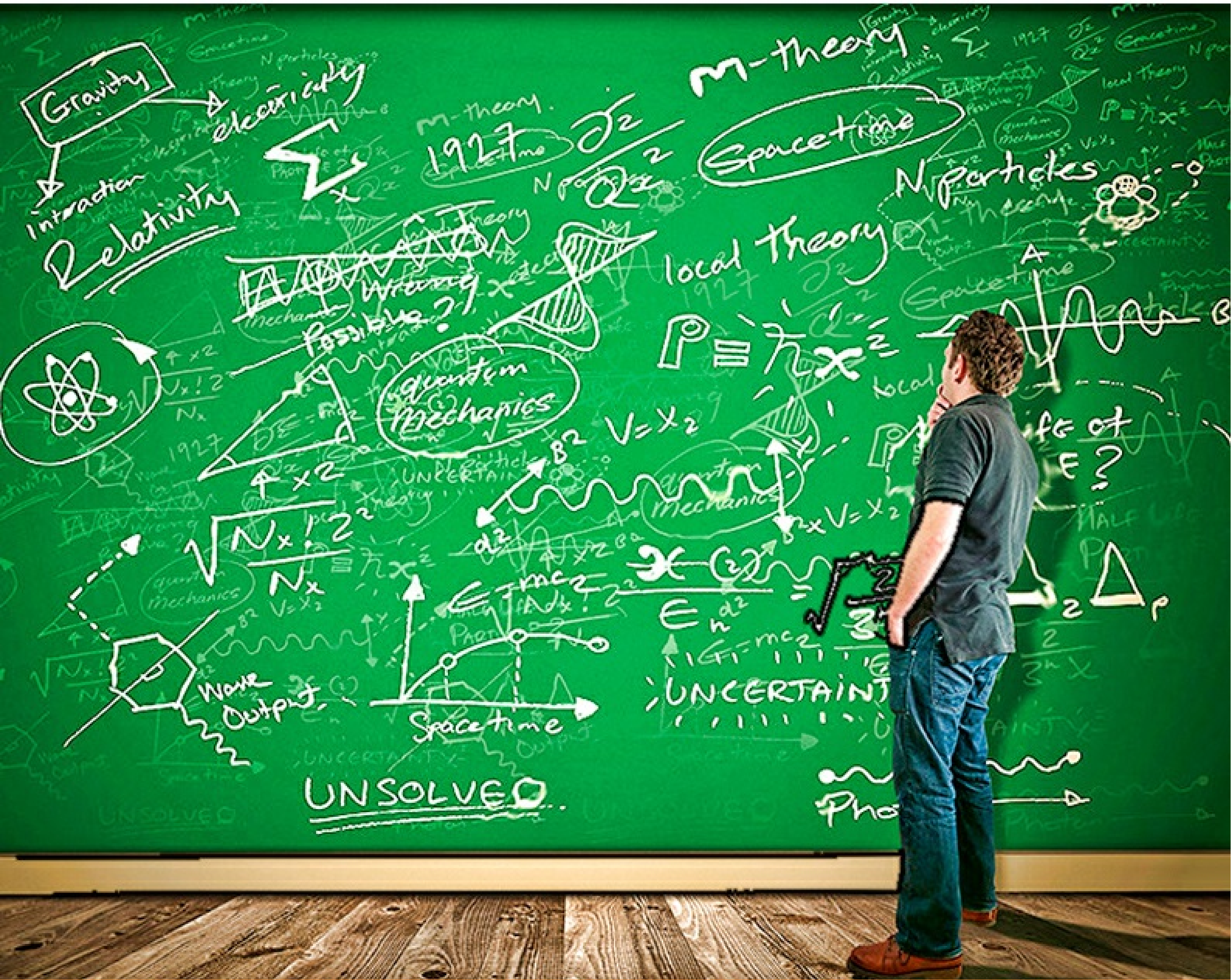
Ulrich Zürcher

Algebra-Based College Physics: Part I

Mechanics to Thermal Physics



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Mechanics to Thermal Physics

Algebra-Based College Physics: Part I – Mechanics to Thermal Physics
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Contents

Forword

Introductory physics provides a comprehensive overview of basic principles with the goal to find solutions to *numerical* problems: 'How long does it take for a ball to hit the ground after being dropped from the top of 10-m tower,' 'What is the acceleration of a falling YoYo,' etc. The solutions require that the relevant physical concepts are identified and then the appropriate equations are written down, manipulated, and solved.

Students' understanding of difficult concepts is improved when they are exposed to several different views of the same problem, and physics is no exception. It is particularly helpful when students are exposed to context-rich problems, since it allows them to make connections to other science disciplines. This is the role of lectures and recitations,and the lecturer should adapt his illustrative examples to his/her audience [eg., premeds, engineers, …]. The accompanying text should be geared towards self-study and focus on main concepts.

This is the guiding principle of the present text; we follow the outline of a standard introductory physics text. We explain the key ideas of each chapter, which are then illustrated by solving one or two 'typical' problems. This keeps the text at a reasonable length and yet is still comprehensive. We use topics covered on the MCAT as a general guiding principle for the selection of topics, as well as the overall depth and breadth of the text.

We use different fonts to distinguish between explanation [in Serif font] and examples [in Sans Serif].

1 Mathematical Tools

1.1 Units

Physical quantities are measured using instruments; a meter stick for length L, a clock for time T, and a balance scale for mass m. Thus, physical quantities have *magnitudes* and *units*. The fundamental units in the international System of Units [SI] are:

Quantity	Unit
Length	meter $[L] = \mathrm{m}$
Time	second $[T] = \mathrm{s}$
Mass	kilogram $[m] = \mathrm{kg}$

For example, we write for distance $d = 4.3\,\mathrm{m}$, for time $t = 76.4\,\mathrm{s}$, or for mass $m = 0.56\,\mathrm{kg}$. Prefixes are used for small and large values:

Prefix	Symbol	Factor
Giga	G	10^9
Mega	M	10^6
kilo	k	10^3
centi	c	10^{-2}
milli	m	10^{-3}
micro	μ	10^{-6}
nano	n	10^{-9}
pico	p	10^{-12}

Thus, $3.4\,\mathrm{nm} = 3.4 \times 10^{-9}\,\mathrm{m}$ and $0.34\,\mu\mathrm{s} = 3.4 \times 10^{-7}\mathrm{s}$.

In any equation, the units on the left and right side must be equal: *dimensional analysis*. For example, we have distance $[L]$, time $[T]$, and speed $[L/T]$. We use dimensional analysis to check whether an equation is possibly correct. For the formula $x = \frac{1}{2}vt^2$, we have $[L] \stackrel{?}{=} ([L]/[T])[T]^2 = [L][T]$. That is, equation $x = \frac{1}{2}vt^2$ is incorrect. For $x = \frac{1}{2}vt$ we get $[L] \stackrel{?}{=} ([L]/[T])[T] = [L]$, so that the equation $x = \frac{1}{2}vt$ might be correct. Dimensional analysis does not tell you anything about factors such as $\frac{1}{2}$, π, etc.

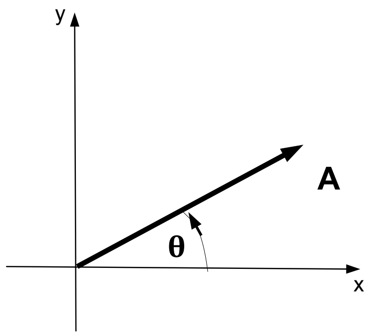
The argument of mathematical functions [sine, cosine, the other trigonometric functions, exponential function, logarithm, etc] are always dimensionless. Thus, for the equation $x = A\cos Bt$, we have $[Bt] = 1$ so that $[B] = 1/[t] = 1/\mathrm{s}$, and $[A] = [x] = \mathrm{m}$.

1.2 Scalars and Vectors

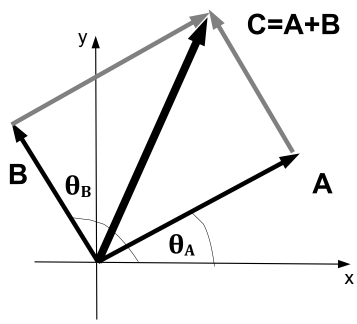
Some physical quantities are fully specified by magnitude [and units]: "meet me at 12:34 pm," "the distance between two cities is 287 km," "today's high temperature is 32°C," and others. Such quantities are called *scalars*. On the other hand, some quantities have both magnitude [and units] and direction. If you are lost in the woods, the advice "you find the hut when you walk 3.3 km" is no help since direction is also needed: "you find the hut when you walk 3.3 km to the south." The quantity of distance *plus* direction is called *displacement*.

There are two different methods of characterizing displacement.

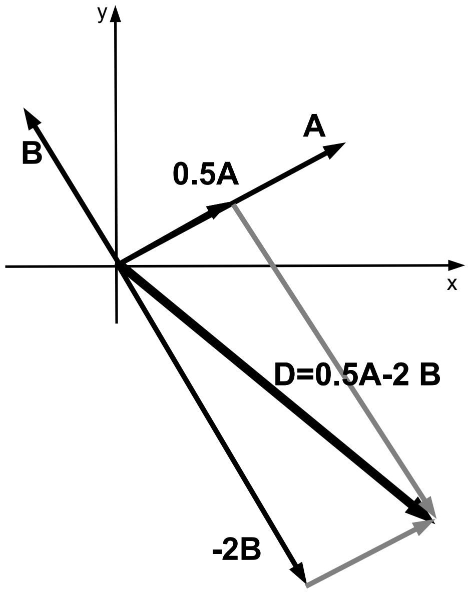
1. Girl Scout method: walk 1.2 mi in NNE direction.
   Two dimensional coordinate system: angle θ from positive x -axis in counterclockwise-direction



Vector addition: the sum of the two vectors is defined by connecting the head of the first vector to the tail of the second vector ["head-to-tail"]. That is, the vectors $\vec{A}$ and $\vec{B}$ are the sides of a parallelogram, and the sum $\vec{C} = \vec{A} + \vec{B}$ is the diagonal:



Vectors can be multiplied by a scalar: the vector $c\vec{A}$ has the magnitude cA and the same direction as the vector $\vec{A}$. If the scalar is negative, $c < 0$, the vector $c\vec{A}$ has the magnitude $|c|A$ and is pointed in the opposite direction: $\theta \to \theta + 180°$. Subtraction of two vectors is defined by $\vec{D} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B})$.



2. City-dweller. In a city such as NYC, the street grid ["blocks"] provides the fundamental unit to describe the location: "Second Avenue and 34th Street." Correspondingly, the displacement can be described by the number of blocks in east-west and north-south direction: "walk two blocks east and three blocks south." We define components:

$$A_x = A\cos\theta, \tag{1}$$

$$A_y = A\sin\theta, \tag{2}$$

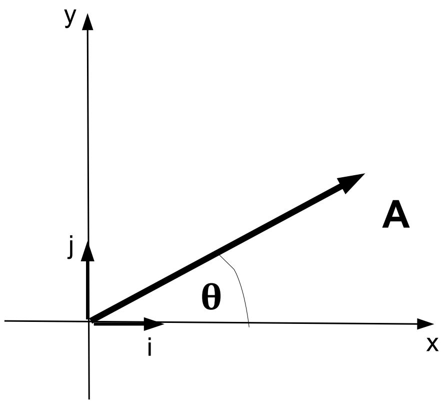
with

$$A^2 = A_x^2 + A_y^2. \tag{3}$$

and

$$\tan\theta = \frac{A_y}{A_x}. \tag{4}$$

We then write $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath}$, where $\hat{\imath}$ and $\hat{\jmath}$ are the unit vectors [unit directions] along the x - and y -directions.



In component form, multiplication by a scalar is given by $A'_x = cA_x$ and $A'_y = cA_y$, and vector addition is defined by the addition of the respective components,

$$C_x \ = \ A_x + B_x, \tag{5}$$

$$C_y \ = \ A_y + B_y. \tag{6}$$

Example: Given two vectors in component form $A_x = 7$, $A_y = 4$ and $B_x = -3$, and $B_y = +4$, find the magnitude and direction of the vector $\vec{D} = 0.5\,\vec{A} - 2\vec{B}$.

Solution: We have the components of the vctor $\vec{D}$:

$$D_x \ = \ 0.5A_x + (-2)B_x = 0.5 \cdot 7 - 2 \cdot (-3) = 9.5,$$

$$D_y \ = \ 0.5A_y + (-2)B_y = 0.5 \cdot 4 - 2 \cdot 4 = -6.$$

We find the magnitude,

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{9.5^2 + (-6)^2} = \sqrt{126.25} = 11.2,$$

and the direction,

$$\tan\theta_D = \frac{D_y}{D_x} = \frac{-6}{9.5} = -0.63 \qquad \longrightarrow \qquad \theta_D = 32.3^\circ.$$

2 Motion in 1 -Dimension

The origin of *Natural Philosophy* in ancient [Greek] times can be traced back trying to understand the motion of objects. In modern-day physics, the analysis of motion is separated into two distinct parts: (1) the description of the motion [*kinematics*] and (2) the cause of the motion [*dynamics*]. The discussion of kinematics is separated into two parts: motion in one spatial dimension and motion in two dimensions. Motion in more than two dimensions does not add more of a physics context.

2.1 Displacement and Distance

In one dimension, the position of an object is described by the coordinate x. The goal of kinematics is to find the position as a function of time $x = x(t)$. We define displacement as the vector from the initial position to the final position:

$$\Delta x = x_f - x_i. \tag{7}$$

In one dimension, the vector character of displacement is reflected by the fact that $\Delta x > 0$ or $\Delta x < 0$. If the coordinate is along East-West, then $\Delta x > 0$ means traveling east and $\Delta x < 0$ means traveling west.

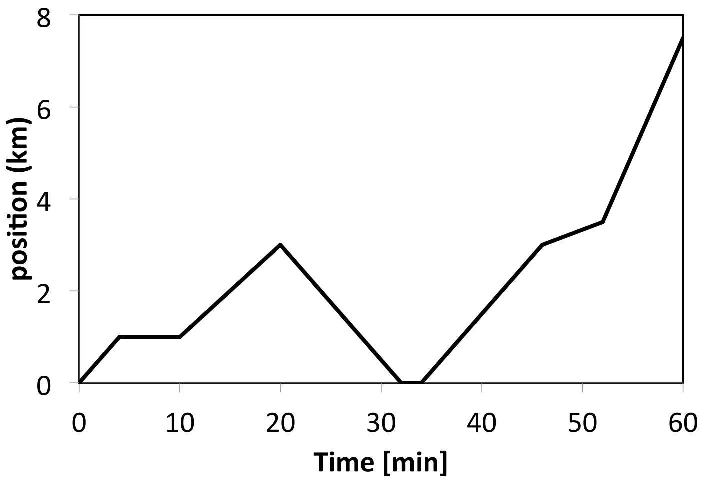
Driving in a city

- Drive 1 km to *Cafe Newton* in 4 min.
- Go inside *Cafe Newton* and wait 6 min.
- Drive 2 km to the intersection of *Copernicus/Kepler Roads* in 10 min.
- Realize that the computer was left at home, drive home in 12 min. Get laptop in 2 min
- Drive back to *Copernicus/Kepler* intersection in 12 min.
- Drive 0.5 km to *Galilei Circle* in 6 min.
- Drive 4 km along *Hubble Highway* in 8 min.

The description of the motion in a table:

t [min]	0	4	10	20	32	34	46	52	60
x [km]	0	1	1	3	0	0	3	3.5	7.5

We represent this in a graph of position x vs. time t .



The displacement is $\Delta x = 3\,\mathrm{km}$ between $t_i = 0$ and $t_f = 20\,\mathrm{min}$, it is $\Delta x = -3\,\mathrm{km}$ between $t_i = 20\,\mathrm{min}$ and $t_f = 32\,\mathrm{min}$, and it is $\Delta x = 7.5\,\mathrm{km}$ between $t_i = 0$ and $t_f = 60\,\mathrm{min}$.

The odometer in a car measures the traveled distance. The distance traveled is $3\,\mathrm{km}$ between $t_i = 0$ and $t_f = 20\,\mathrm{min}$, it is $3\,\mathrm{km}$ between $t_i = 20\,\mathrm{min}$ and $t_f = 32\,\mathrm{min}$, and it is $7.5\,\mathrm{km}$ between $t_i = 0$ and $t_f = 60\,\mathrm{min}$.

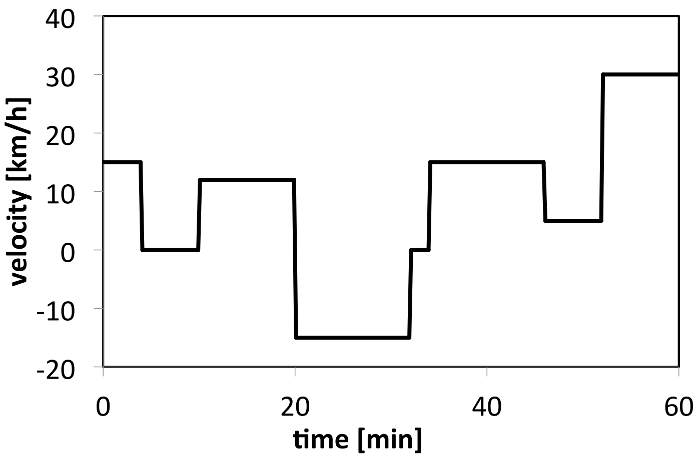
2.2 Velocity and Speed

The graph consists of piecewise straight lines, characterized by an intercept and a slope. The slope is determined by the ratio of the change along the vertical axis Δx, divided by the change along the horizontal Δt ["rise-over-run"]. The (average) velocity is defined by

$$v = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0}, \tag{8}$$

where x_0 is the position at time t_0: $x_0 = x(t_0)$. The unit of velocity is $[v] = [x]/[t] = \mathrm{m/s}$. The average velocities for the above time intervals are:

$[t_i, t_f]$ [min]	[0,4]	[4,10]	[10,20]	[20,32]	[32, 34]	[34 46]	[46,52]	[52, 60]
v [km/h]	15	0	12	-15	0	15	5	30



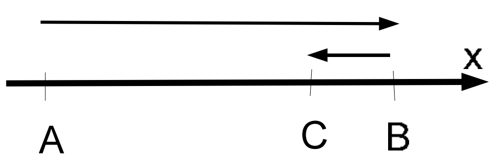
We use $t_0 = 0$ and solve Eq.(8) for the coordinate x when the velocity is constant:

$$x(t) = x_0 + vt. \tag{9}$$

The (average) speed is defined as the ratio of traveled distance divided by the change in time:

$$\text{speed} = \frac{\text{distance traveled}}{\Delta t}. \tag{10}$$

Note that there is no equation for distance traveled and speed.



Example 1: Emmy starts at time $t = 0$ and drives along a straight path: she starts at point A, drives straight to point B, turns aorund and drives to point C. She ends up at the point at time $t = 14.7\,\mathrm{s}$. Her average speed is 17.0 m/s, and the average velocity is 9.3 m/s. **a)** Find distances between points A and B and points B and C. **b)** Emmy drives twice as fast from A to B than she drives from B to C. Find her respective speeds.

Solution: We have the distance L between A and B and the distance d between B and C: then

$$\begin{aligned} L + d &= 17.0\,\frac{\mathrm{m}}{\mathrm{s}} \cdot 14.7\,\mathrm{s} = 250.0\,\mathrm{m}, \\ L - d &= 9.3\,\frac{\mathrm{m}}{\mathrm{s}} \cdot 14.7\,\mathrm{s} = 136.7\,\mathrm{m}. \end{aligned}$$

We get

$$L = \frac{250.0\,\mathrm{m} + 136.7\,\mathrm{m}}{2} = 193.4\,\mathrm{m}, \quad d = \frac{250.0\,\mathrm{m} - 136.7\,\mathrm{m}}{2} = 56.6\,\mathrm{m}.$$

For times $t_{AB} = L/v_{AB}$ and $t_{BC} = d/v_{BC}$, where $v_{BC} = v_{AB}/2$. Since $t_{AB} + t_{BC} = 14.7\,\mathrm{s}$, we get,

$$\frac{193.4\,\mathrm{m}}{v_{AB}} + \frac{56.7\,\mathrm{m}}{v_{BC}} = \frac{193.4\,\mathrm{m}}{v_{AB}} + \frac{2 \cdot 56.7\,\mathrm{m}}{v_{AB}} = 14.7\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We get

$$v_{AB} = \frac{193.4\,\mathrm{m} + 2\cdot 56.7\,\mathrm{m}}{14.7\,\mathrm{s}} = 20.9\,\frac{\mathrm{m}}{\mathrm{s}},$$

and $v_{BC} = 10.4\,\mathrm{m/s}$.

Example 2: A jaguar can reach speeds of 30 m/s. The fastest person is capable of reaching a speed of 10 m/s. Suppose the the person and the jaguar are 500 m apart. Assume that they are both constantly at their top speed. How long does it take the jaguar to catch up to the person?

Solution: We choose a coordinate system with $x = 0$ at the person's starting position so that $x = 500\,\mathrm{m}$ for the initial position of the jaguar. We find the time-dependent coordinates of the person and jaguar:

$$x_p(t) \;=\; 0 - 10\,\frac{\mathrm{m}}{\mathrm{s}}\cdot t, \qquad \text{(person)},$$

$$x_j(t) \;=\; 500\,\mathrm{m} - 30\,\frac{\mathrm{m}}{\mathrm{s}}\cdot t, \qquad \text{(jaguar)}.$$

The jaguar catches up to the man when $x_p = x_j$ so that

$$-10\,\frac{\mathrm{m}}{\mathrm{s}}\cdot t = 500\,\mathrm{m} - 30\,\frac{\mathrm{m}}{\mathrm{s}}\cdot t \quad\longrightarrow\quad t^* = \frac{500\,\mathrm{m}}{20\,\mathrm{m/s}} = 25.0\,\mathrm{s}.$$

Discussion: Key to solving kinematics problems is often setting up equations for the coordinates of one or more objects [e.g., the person and the jaguar], and then finding an equation for a particular event [e.g., jaguar catching up to the person].

2.3 Acceleration

In general, the velocity varies with time; mathematically, we say that the velocity is a function of time $v = v(t)$. This is called *instantaneous* velocity. The rate of change in velocity is acceleration:

$$a = \frac{\Delta v}{\Delta t} = \frac{v - v_0}{t - t_0}, \tag{11}$$

where v is the velocity at time t, and v_0 is the velocity at time t_0. The unit of acceleration is $[a] = [v]/[t] = \mathrm{m/s}^2$. We set $t_0 = 0$, and find,

$$v = v_0 + at, \qquad \text{(instantaneous velocity)}. \tag{12}$$

We only consider the case when the acceleration is piecewise constant. The acceleration can be positive or negative. The object speeds up, when both velocity and acceleration are positive or negative; and the object slows down, when the velocity is positive and the acceleration is negative [or vice versa].

The average velocity during the time interval $[0,t]$ follows,

$$\bar{v} = \frac{1}{2}[v+v_0] = v_0 + \frac{1}{2}at, \qquad \text{(average velocity)}. \tag{13}$$

Note that the RHS of Eq. (13) is equal to the instantantaneous velocity at time $t/2$: $v(t/2) = v_0 + a(t/2)$. This is called the *midpoint rule*. The displacement follows,

$$x - x_0 = \bar{v}t = \left(v_0 + \frac{1}{2}at\right) \cdot t = v_0 t + \frac{a}{2}t^2. \tag{14}$$

We find an equation that relates displacement to velocity by eliminating the time using Eq. (12), $t = (v - v_0)/a$, so that $\Delta x = x - x_0 = (v_0 + v)/2 \cdot (v - v_0)/a$ so that

$$v^2 = v_0^2 + 2a\Delta x. \tag{15}$$

Eqs. (12)-(15) are the basis for motion with constant acceleration. We will always assume that the acceleration is piecewise constant, i.e., is constant during finite time intervals.

Example 3: A car is accelerating at $a = 0.8\,\mathrm{m/s}^2$. It passes through a 25.0-m wide crossing in a time of $\Delta t = 4.6\,\mathrm{s}$. Assume that the car starts from rest. Calculate the time to reach the crossing and the distance from the crossing.

Solution: We have for the average speed through the crossing,

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{25.0\,\mathrm{m}}{4.6\,\mathrm{m/s}} = 5.4\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We set $t = 0$ and $t = t_1$ when the car enters and leaves the crossing, respectively. We write v_0 for the speed when the locomotive enters the crossing. Now we use the midpoint rule,

$$v_{\text{ave}} = v(t/2) \quad \longrightarrow \quad 5.4\,\frac{\mathrm{m}}{\mathrm{s}} = v_0 + a \cdot \frac{t_1}{2} = v_0 + 0.8\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 2.3\,\mathrm{s} \quad \longrightarrow \quad v_0 = 3.6\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The car starts at $t = -t^*$. We then have

$$v_0 = a|t^*| \quad \longrightarrow \quad |t^*| = \frac{v_0}{a} = \frac{3.6\,\mathrm{m/s}}{0.8\,\mathrm{m/s}^2} = 4.5\,\mathrm{s},$$

and for the distance between the start and the crossing:

$$d = \frac{a}{2}|t^*|^2 = \frac{0.8\,\mathrm{m/s}^2}{2}\,(4.5\,\mathrm{s})^2 = 8.1\,\mathrm{m}.$$

Alternatively, the distance follows from $v_0^2 = 2ad$, or $d = v_0^2/2a = (3.6\,\mathrm{m/s})^2/(2 \cdot 0.8\,\mathrm{m/s}^2) = 8.1\,\mathrm{m}$.

Example 4: A dog and her handler are at rest, and are facing each other at a distance of 40.0 m. On command, they run towards each other: the handler runs at a constant speed $v_\mathrm{p} = 2.5\,\mathrm{m/s}$, while the dog first runs with constant acceleration $a_\mathrm{d} = 2.0\,\mathrm{m/s}^2$ until she reaches her maximum speed of $v_\mathrm{d} = 4.5\,\mathrm{m/s}$. **a)** Calculate the time it takes the dog to reach her maximum speed. Choose $t = 0$ when the handler and dog start to run. **b)** Find the positions of the handler and dog at the instant the dog reaches her maximum speed. Choose the coordinate, such as that the handler is at the origin $x = 0$ at time $t = 0$. **c)** Find the (common) position where the handler and dog meet.

Solution: Since $v = v_0 + at$ with $v_0 = 0$, we have for the time,

$$t_\mathrm{d} = \frac{v_\mathrm{d}}{a_\mathrm{d}} = \frac{4.5\,\mathrm{m/s}}{2.0\,\mathrm{m/s}^2} = 2.25\,\mathrm{s},$$

and for the dog's coordinate,

$$x_\mathrm{d} = x_{0,\mathrm{d}} - \frac{1}{2} a_\mathrm{d}\, t_\mathrm{d}^2 = 40.0\,\mathrm{m} - \frac{1}{2} \cdot 2.0 \frac{\mathrm{m}}{\mathrm{s}^2} \cdot (2.25\,\mathrm{s})^2 = 34.9\,\mathrm{m},$$

for the handler,

$$x_\mathrm{p} = x_{0,\mathrm{p}} + v_\mathrm{p} t_\mathrm{d} = 2.5\,\frac{\mathrm{m}}{\mathrm{s}} \cdot 2.25\,\mathrm{s} = 5.6\,\mathrm{m}.$$

We write $t = 2.25\,\mathrm{s} + t'$ for times $t > t_\mathrm{d}$. We then have the coordinates of the dog and her handler for times $t' > 0$:

$$x_\mathrm{d}(t') \;=\; 34.9\,\mathrm{m} - 4.5\,\frac{\mathrm{m}}{\mathrm{s}} \cdot t',$$

$$x_\mathrm{p}(t') \;=\; 5.6\,\mathrm{m} + 2.5\,\frac{\mathrm{m}}{\mathrm{s}} \cdot t'.$$

Set $x_\mathrm{d}(t') = x_\mathrm{p}(t')$ so that

$$34.9\,\mathrm{m} - 4.5\,\frac{\mathrm{m}}{\mathrm{s}} \cdot t' = 5.6\,\mathrm{m} + 2.5\,\frac{\mathrm{m}}{\mathrm{s}} \cdot t'.$$

Rearrange

$$29.3\,\mathrm{m} = 7.0\,\frac{\mathrm{m}}{\mathrm{s}} \cdot t \quad \longrightarrow \quad t' = 4.2\,\mathrm{s}.$$

Plug the time t' into either $x_\mathrm{d}(t')$ or $x_\mathrm{p}(t')$, and find $x_\mathrm{d} = x_\mathrm{p} = 16.1\,\mathrm{m}$.

2.4 Free Fall

Free fall is an important case of motion with constant acceleration. All objects near the Earth's surface fall with the same acceleration, if effects due to air resistance can be eliminated or ignored. We choose the $+y$ – axis upwards, so that $v > 0$ and $v < 0$ describes an object flying upwards and downwards, respectively. Then $a_y = -g$ [negative sign indicates downwards]. We then have the kinematics equations for free fall,

$$v(t) \;=\; v_0 - gt, \tag{16}$$

$$y(t) \;=\; y_0 + v_0 t - \frac{1}{2} g t^2, \tag{17}$$

$$v^2 \;=\; v_0^2 - 2g\Delta y. \tag{18}$$

The speed of the object decreases on the way up [$v > 0$ and $a = -g < 0$], and increases on the way down [$v < 0$ and $a = -g < 0$]; the instantaneous velocity is zero at the highest point.

Example 5: A ball is dropped from the roof of a tower with a height of 81 m. **a)** How long will it take to hit the ground? **b)** Calculate the speed without calculating the time.

Solution: We choose $y = 0$ at the ground. Then $y_0 = 81\,\mathrm{m}$ and $v_0 = 0$, so that

$$y = 0 = 81\,\mathrm{m} - \frac{1}{2} g (t^*)^2 \quad \longrightarrow \quad t^* = \sqrt{\frac{2 \cdot 81\,\mathrm{m}}{9.8\,\mathrm{m/s}^2}} \simeq 4.0\,\mathrm{s}.$$

We get for the speed right before hitting the ground,

$$v_f = v_0 - gt = -9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 4.0\,\frac{\mathrm{m}}{\mathrm{s}}.$$

A negative sign means that the object falls downwards.

We have $\Delta y = y_f - y_0 = 0 - 81\,\mathrm{m} = -81\,\mathrm{m}$. Then:

$$v_f^2 = \left(0\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - 2\cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot(-81\,\mathrm{m}) \simeq +1600\,\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2 \quad\longrightarrow\quad |v_f| = 40\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The second method does not give the sign [positive or negative] of the velocity.

Discussion: Rather than calculating the time from the vertical displacement Δy by solving a quadratic equation, it is always possible to find the time in two steps: (1) find the speed using Eq. (18), and add the appropriate sign to find the velocity, and (2) find the time using Eq. (16).

Example 6: A ball is thrown along the vertical. The coordinate of the ball is observed as a function of time $y(t)$. We choose $y = 0$ at the ground. The ball is at $y = 13.2\,\mathrm{m}$ at time $t = 0$, and at $y = 5.0\,\mathrm{m}$ at time $t = 3.2\,\mathrm{s}$. **a)** Find the displacement and average velocity of the ball between $t = 0$ and $t = 3.2\,\mathrm{s}$. **b)** Find the initial and final velocity of the ball. **c)** What is the height of the ball's peak above ground? **d)** What is the average speed of the ball between $t = 0$ and $t = 3.2\,\mathrm{s}$?

Solution: We have for the displacement $\Delta y = y - y_0 = 5.0\,\mathrm{m} - 13.2\,\mathrm{m} = -8.2\,\mathrm{m}$, and for the average velocity

$$v_{\mathrm{max}} = \frac{\Delta y}{\Delta t} = \frac{-8.2\,\mathrm{m}}{3.2\,\mathrm{s}} = -2.6\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We use $t = 0$ when the ball is launched upward. We use the midpoint rule to find the instantaneous velocity at $t_{\mathrm{mid}} = (0 + 3.2\,\mathrm{s})/2 = 1.6\,\mathrm{s}$,

$$v(1.6\,\mathrm{s}) = -2.6\,\frac{\mathrm{m}}{\mathrm{s}} = v_0 - 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 1.6\,\mathrm{s} \quad\longrightarrow\quad v_0 = -2.6\,\frac{\mathrm{m}}{\mathrm{s}} + 15.7\,\frac{\mathrm{m}}{\mathrm{s}} = 13.1\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We get for the final velocity at $t = 3.2\,\mathrm{s}$,

$$v_f = 13.1\,\frac{\mathrm{m}}{\mathrm{s}} - 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 3.2\,\mathrm{s} = -18.3\frac{\mathrm{m}}{\mathrm{s}^2}.$$

We have $v = 0$ at the highest point. We find the displacement to the peak,

$$v^2 = 0 = v_0^2 - 2g\Delta y \quad\longrightarrow\quad \Delta y = \frac{v_0^2}{2g} = \frac{(13.1\,\mathrm{m/s})^2}{2\cdot 9.8\,\mathrm{m/s}^2} = 8.8\,\mathrm{m}.$$

We have for the $y_{\text{peak}} = 13.2\,\text{m} + 8.8\,\text{m} = 22.0\,\text{m}$, and the total distance traveled,

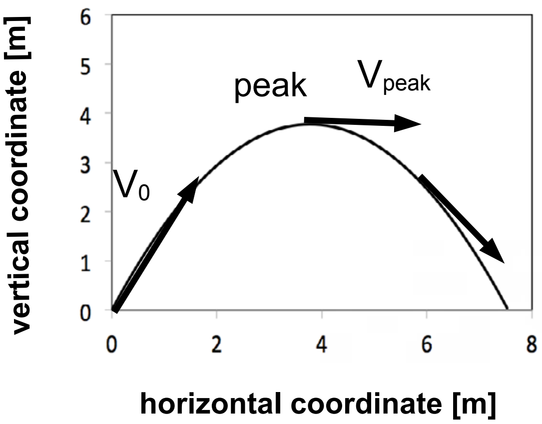
$$d = (22.0\,\text{m} - 13.2\,\text{m}) + |5.0\,\text{m} - 22.0\,\text{m}| = 25.8\,\text{m}$$

so that for the average speed,

$$\text{speed} = \frac{25.8\,\text{m}}{3.2\,\text{s}} = 8.1\,\frac{\text{m}}{\text{s}}.$$

3 Motion in 2 Dimensions: Projectile Motion

A person walks at a speed v_p along the horizontal. The person throws a ball up in the air with velocity v_b , and easily catches it while walking. For this person, it appears that the ball undergoes free-fall motion. On the other hand, for an observer at rest, the ball undergoes motion along both the horizontal and vertical axis. The trajectory of the ball as seen by the observer at rest is an 'arc:'projectile motion. This would be the same if the ball is thrown at the initial speed $v_0 = \sqrt{v_p^2 + v_b^2}$ at the angle $\tan\theta_0 = v_b/v_p$ above the horizontal.



The shape of the trajectory is an inverted parabola.

These everyday experiments show that the motion of the ball along the horizontal and vertical directions are independent of each other. The velocity component along the horizontal is constant [and thus the acceleration is zero], while the velocity component along the vertical follows laws of free fall. In particular, the velocity vector at the highest point ["peak"] is directed along the horizontal, $\vec{v}_{\rm peak} = v_p\hat{\imath}$, since the peak is the turning point for the motion along the vertical $v_{y,\rm peak} = 0$. The instantaneous speed is equal to the magnitude of the instantaneous velocity vector $|\vec{v}|$. The speed of the object in projectile motion decreases on the way up, and increases on the way down.

The acceleration is constant and directed downwards,

$$a_x = 0, \quad (19)$$

$$a_y = -g = -9.8\,\frac{\rm m}{\rm s^2}, \quad (20)$$

for the components of the velocity vector,

$$v_x(t) = v_{x,0} = \text{const}, \tag{21}$$

$$v_y(t) = v_{y,0} - gt, \tag{22}$$

and the coordinates,

$$x(t) = x_0 + v_{x,0}t, \tag{23}$$

$$y(t) = y_0 + v_{y,0}t - \frac{1}{2}gt^2. \tag{24}$$

Here, we use the usual convention that $v_y > 0$ means "going up," $v_y < 0$ means "going down" and $v_x > 0$ means "going to the right," and $v_x < 0$ means "going to the left."

In Eqs. (20)-(24), the trajectory is determined by the components of the initial velocity along the x - and y -coordinates. Instead of the two components of the velocity, one can also specify the speed $v_0 = \sqrt{v_{x,0}^2 + v_{y,0}^2}$ and the direction $\tan\theta = v_{y,0}/v_{x,0}$. While the initial speed and launch angle at time $t = 0$ is a 'popular choice,' more general conditions are possible. The trajectory is fully determined by *two* independent quantities (v_x, v_y, x, y) at one or two times t_1 and t_2, although the necessary algebra might become cumbersome. Solving problems starts by determing whether it is necessary to find a particular time t, and if so, how that time is found. There are three possibilities: using the vertical velocity v_y, the horizontal displacement Δx, or the vertical displacement Δy.

Example 1: A projectile is launched with $v_0 = 19.2\,\mathrm{m/s}$ at angle $\theta_0 = 38.7°$. **a)** Does the projectile 'clear' a 3.0 m- high wall that is 30 m from the launch? **b)** How far behind the wall does the projectile hit the ground?

Solution: We calculate the components of the initial velocity vector:

$$v_{x,0} = 19.2\,\frac{\mathrm{m}}{\mathrm{s}}\,\cos 38.7° = 15\,\frac{\mathrm{m}}{\mathrm{s}},$$

$$v_{y,o} = 19.2\,\frac{\mathrm{m}}{\mathrm{s}}\,\sin 38.7° = 12\,\frac{\mathrm{m}}{\mathrm{s}}.$$

For this problem, we need to find the time when the projectile clears [*or hits*] the wall. That is, we set $\Delta x = 30.0\,\mathrm{m}$ so that for the time $t = t^*$:

$$t^* = \frac{\Delta x}{v_{x,0}} = \frac{30\,\mathrm{m}}{15\,\mathrm{m/s}} = 2.0\,\mathrm{s}.$$

We then find the vertical coordinate of the projectile at that instant,

$$y = 12\,\frac{\mathrm{m}}{\mathrm{s}} \cdot 2.0\,\mathrm{s} - \frac{1}{2} 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot (2\,\mathrm{s})^2 = 24\,\mathrm{m} - 20\,\mathrm{m} = 4\,\mathrm{m}.$$

Since $y > 3\,\mathrm{m}$, the projectile clears the wall. The projectile falls to the ground when $y = 0$ so that

$$y = v_{0,y} t - \frac{g}{2} t^2 = \left(v_{0,y} - \frac{gt}{2} \right) t \quad \longrightarrow t' = 2\frac{v_{0,y}}{g} = 2\,\frac{12\,\mathrm{m/s}}{9.8\,\mathrm{m/s}^2} = 2.45\,\mathrm{s}.$$

We thus get for the horizontal displacement,

$$R = v_{x,0} t' = 15\,\frac{\mathrm{m}}{\mathrm{s}} \cdot 2.45\,\mathrm{s} = 36.8\,\mathrm{m}.$$

That is, the projectile falls to the ground 6.8 m behind the wall.

Discussion: The time to reach the peak follows from $v_y = v_{y,0} - gt = 0$ or $t_{\text{peak}} = v_{y,0}/g$. Thus, the time for the projectile to hit the ground is twice that of the projectile to reach the peak: this property reflects the symmetry of the trajectory.

The quantity R is referred to as the *range*. We find for the time to reach the peak $v_{\text{peak}} = v_0 \sin\theta_0/g$, so that for the range $R = v_0\cos\theta_0 \cdot 2v_0 \sin\theta_0/g$, or,

$$R = \frac{v_0^2}{g}\sin 2\theta_0, \tag{25}$$

where we use $2\sin\theta_0\cos\theta_0 = \sin 2\theta_0$. This shows that the maximum range projectile is achieved when $\theta_0 = 45°$.

Example 2: A device is hidden inside a ball, enabling you to measure the speed of the ball but not its direction. You forget to write down the times at which the ball was flying with different speeds. The ball undergoes *projectile motion* in the Earth's gravitational field. **a)** From your records, you determine that the maximum speed of the ball was $30.0\,\mathrm{m/s}$, and that the minimum speed was $18.0\,\mathrm{m/s}$. Find the direction in which the ball was launched. **b)** How long is the ball in air until it hits the ground? **c)** What distance along the ground does the ball travel through the air until it hits the ground?

Solution: We have $v_x = v_{0,x} = v_{\min} = 18.0\,\mathrm{m/s}$. Since $v_0 = v_{\max} = 30.0\,\mathrm{m/s}$, we get:

$$v_{0x} = v_0\cos\theta_0 \quad \longrightarrow \quad \cos\theta_0 = \frac{v_{0x}}{v_0} = \frac{18.0\,\mathrm{m/s}}{30.0\,\mathrm{m/s}} = 0.6.$$

For the angle, this gives $\theta_0 = 53.1°$. We calculate the vertical component of the initial velocity:

$$v_{0,y} = \sqrt{v_0^2 - v_{0,x}^2} = \sqrt{\left(30.0\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - \left(18.0\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2} = 24.0\,\frac{\mathrm{m}}{\mathrm{s}}.$$

Now we calculate the time to reach the highest point,

$$v_y = 0 = v_{0,y} - gt^* \quad \longrightarrow \quad t^* = \frac{v_{0,y}}{g} = \frac{24.0\,\mathrm{m/s}}{9.8\,\mathrm{m/s}^2} = 2.45\,\mathrm{s}.$$

The total time in the air follows $t_{\text{tot}} = 2t^* = 4.9\,\mathrm{s}$. We have for the range,

$$R = \Delta x_{\max} = v_x t_{\text{tot}} = 18.0\,\frac{\mathrm{m}}{\mathrm{s}} \cdot 4.9\,\mathrm{s} = 88.2\,\mathrm{m}.$$

4 Dynamics: Newton's Laws

Book on Table: The book moves when we push it: here push means that we exert a force. The 'harder' the push, i.e., the greater the force, the faster the book moves. Furthermore, the book moves in the direction of the push. Thus, the velocity is proportional to the force,

$$\vec{F} = A\vec{v}, \qquad \text{(book on table)},$$

for some constant A that characterizes the book and table surface.

Ice skater: The ice skater keeps moving, even when no force acts on the person: force and velocity are no longer proportional to each other. A force acts on the ice skater during the short time when she is pushed by her friend. During that same time interval, she is experiencing an acceleration, since the velocity changes with time. After that, she does not experience any acceleration. Thus, the acceleration is proportional to the force:

$$\vec{F} = m\vec{a}, \qquad \text{(ice skater)},$$

where m is the mass of the ice skater. In dynamics, we write the cause [force] and the effect [motion] on the left-and right-hand side of the equation, respectively. Newton's law $F = ma$ often seems to go against common sense: in many cases, these misconceptions can be resolved by noting that everyday experience [such as driving a car, etc] take place when friction is dominant.

4.1 Newton's Laws

First law: object remains in a state of rest or in a state with constant velocity, unless compelled to change by a net force. The first law defines *inertial reference frames*.

Second law: the acceleration of an object with mass m is proportional to the net force acting on it:

$$\sum \vec{F} = m\vec{a}. \tag{26}$$

Third law: whenever the first body exerts force on the second body, the second body exerts force on the first body that is equal in magnitude and opposite in direction. In short, forces always appear in pairs. The unit for force follows: $[F] = [m] \cdot [a] = \mathrm{kg} \cdot \mathrm{m/s}^2 = \mathrm{N}$, or Newton. The net force is the sum of *physical* forces acting on an object. In introductory physics, the list of physical forces is quite small: weight mg, the tension T in a rope, the contact force, or normal force F_N, and friction force f_k. When an object sits on a block [so that the acceleration is zero], the net force on the block is zero. The weight of the block is directed downwards with magnitude $W = mg$. This force is balanced by the normal force the table exerts on the block; we have for the magnitude $F_N = mg$ and the direction is upwards. "Normal" refers to the mathematical meaning of "perpendicular." In general, the normal force is always directed perpendicular to the surface.

The friction force is directed parallel to the surface. We distinguish two cases: (1) kinetic friction for moving objects. The friction force is directed against the direction of motion, and the magnitude is proportional to the normal force,

$$f_k = \mu_k F_N. \tag{27}$$

When the object is at rest, static friction applies. The magnitude and direction [e.g., left or right] varies, and is determined by the condition that the net force is zero $\sum \vec{F} = 0$. The static friction has a maximum value:

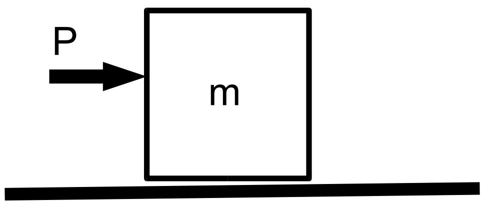
$$f_{s,\max} = \mu_s F_N . \tag{28}$$

The coefficient of static friction is greater than the coeffcient of kinetic friction, $\mu_s > \mu_k$. This explains the fact that a heavy box 'jerks' when it begins to move.

4.2 Free-Body Diagram

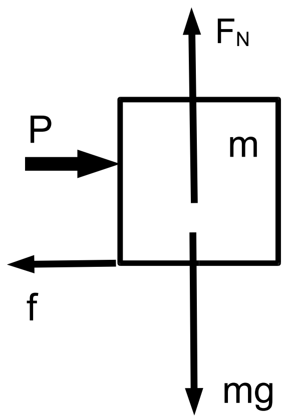
For each object, we replace its interaction of an object with "the rest of the universe" by forces: all forces are drawn in a *free-body diagram*. For example, if an object sits on a surface, the interaction between the object and the table is described by the normal force that the table exerts on the object. If the object is pulled by a rope, the interaction is described by the tension force acting on the object. Newton's second law is written in component form, i.e., for both the x - and y -direction. The same process is repeated for each object of the system of interest. In this way, we find a system of *coupled* equations for the unknown forces and accelerations of the objects.

We discuss four illustrative examples, each of which adds one more element of difficulty. The first problem only deals with a single object; the second problem deals with two objects; the third problem illustrates the "good" choice of a coordinate system; and the fourth problem deals with action-reaction pairs [Newton's third law]. The choice of 'system' depends on the circumstances: we generally exclude the Earth, since it has an enormous mass, and does not move. For astronomical problems, the motion of the Earth must be taken into account.



Example 1: A block with mass $m = 4.0\,\mathrm{kg}$ is sitting on a horizontal surface.

The coefficient of static friction between the block and the table is $\mu_s = 0.32$, and the coefficient of kinetic friction is $\mu_k = 0.29$. Starting from $P = 0$, the push is continuously increased until the block begins to move. **a)** What is the maximum push $P_{\max}$ that can be applied so the block does not move? **b)** After the push is slightly increased, what is the acceleration of the block?



Solution: The forces on the block are the weight mg (down), the normal force F_N (up), the push (right), and the friction force f. The direction of the friction force is then left, because the object is sliding to the right [as a result of the push]. When the block is at rest, we have $f = f_s$. We write Newton's law for the mass m:

$$\sum F_x = P - f_s = 0,$$

$$\sum F_y = F_N - mg.$$

We find $F_N = mg$, so that for the maximum static friction $f_{s,\max} = \mu_s\, mg$:

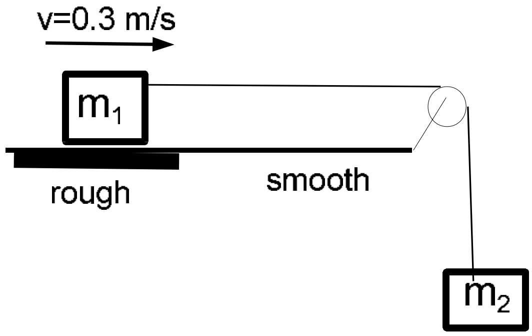
$$P_{\max} = \mu_s\, mg = 0.32 \cdot 4.0\,\mathrm{kg} \cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} = 12.5\,\mathrm{N}.$$

We increase the push by an infinitesimally small amount $P = P_{\max} + \delta P$, so that the block moves and $f = f_k = \mu_k mg = 0.29 \cdot 4.0\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2} = 11.4\,\mathrm{N}$. We then have for the acceleration of the block:

$$\sum F_x = P_{\max} - f_k = ma \quad \longrightarrow \quad a = \frac{P_{\max} - f_k}{m} = \frac{12.5\,\mathrm{N} - 11.4\,\mathrm{N}}{4.0\,\mathrm{kg}} = 0.28\,\frac{\mathrm{m}}{\mathrm{s}^2}.$$

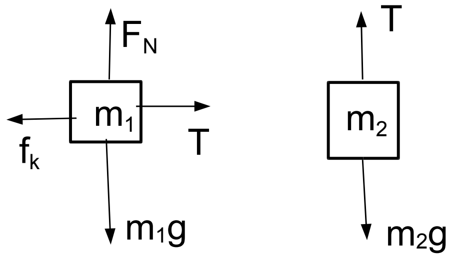
The jump of the acceleration of the block at rest $[a = 0]$ to a non-zero value $[a = 0.28\,\mathrm{m/s^2}]$ is described as a "jerk." After the block moves, we intuitively reduce the push to $P' = f_k = 11.4\,\mathrm{N}$ so that the block is again in mechanical equilibrium $\sum \vec{F} = 0$, and slides at a constant velocity $\vec{v} = \mathrm{const}$.

Discussion: Here F_N and the weight of the block $m\vec{g}$ have the same magnitude, and are directed in opposite directions. However, they are **not** action-reaction forces since they are acting on the same object. When a person holds an object with mass m in her hands, the action-reaction pair is (1) the weight of the object $m\vec{g}$ acting on her hands and (2) the normal force $\vec{F}_N$ that the palm of her hands exerts on the object.



Example 2: A block with mass $m_1 = 2.2\,\mathrm{kg}$ is attached to an unknown mass m_2. When the block m_1 slides across a rough patch [with coefficient of kinetic friction $\mu_k = 0.36$], the velocity of the block is constant $v = 0.3\,\mathrm{m/s}$. **a)** Draw the appropriate freebody diagram(s) for this problem. **b)** Find the unknown mass m_2. **c)** The block m_1 is sliding across a smooth patch of the table. Find the acceleration of the block m_1.

Solution: The forces on m_1 are the weight $m_1 g$ (down), the normal force F_N (up), the tension T (right), and the friction force f_k (left). The forces on m_2 are the weight $m_2 g$ (down) and the tension T (up). The magnitude of the tension forces acting on m_1 and m_2 are the same, because it is a single rope.



We have $\vec{v} = \text{const}$, the acceleration is zero, and the two blocks are in mechanical equilibrium $\sum F = 0$. We write Newton's second law for m_1:

$$\sum F_x = T - f_k = 0 \qquad \sum F_y = F_N - m_1 g = 0,$$

and Newton's second law for m_2:

$$m_2 g - T = 0.$$

Note that we take the direction of the forces into account by appropriate positive and negative signs. This implies that all variables represent the magnitudes of the respective forces, i.e., all variables are positive. This can be used to check the solution of the system of equations. We have $F_N = m_1 g = 21.6\,\text{N}$, so that $f_k = \mu_k F_N = 0.36 \cdot 21.6\,\text{N} = 7.8\,\text{N}$.

We then have $T = f_k$ and

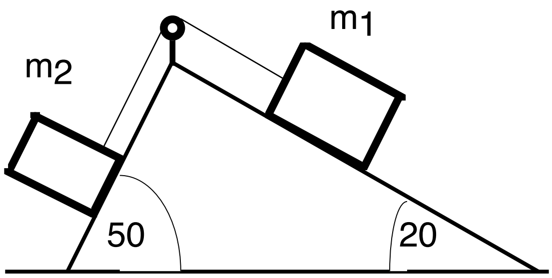
$$m_2 = \frac{T}{m_2} = \frac{f_k}{g} = \frac{7.8\,\text{N}}{9.8\,\text{m/s}^2} = 0.8\,\text{kg}.$$

When the block m_1 is on the smooth table surface, $f_k = 0$. All forces remain the same, although the magnitude of the forces may change. We have

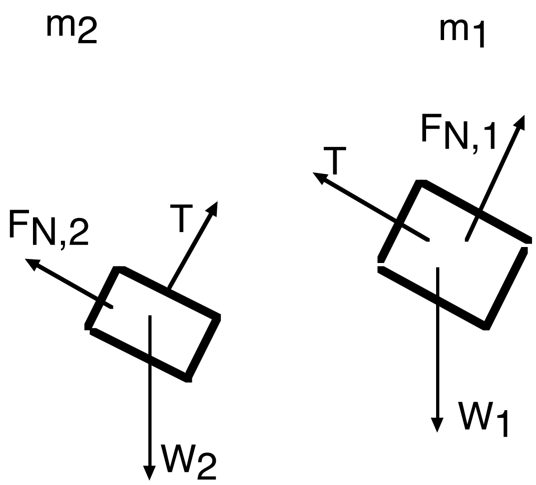
$$T = m_1 a, \qquad m_2 g - T = m_2 a.$$

the common acceleration of the two blocks follows $m_2 g - m_1 a = m_2 a$

$$a = \frac{m_2}{m_1 + m_2} g = \frac{0.8\,\text{kg}}{2.2\,\text{kg} + 0.8\,\text{kg}} \cdot 9.8\,\frac{\text{m}}{\text{s}^2} = 2.6\,\frac{\text{m}}{\text{s}^2}.$$



Example 3: The block 1 with mass $m_1 = 2.5\,\mathrm{kg}$ is moving on a frictionless 20° incline, and the block 2 with mass $m_2 = 4.5\,\mathrm{kg}$ is moving on a frictionless 50° incline. The two blocks are connected by a string. **a)** Draw the appropriate free-body diagram(s) for the problem. **b)** Write down Newton's second law for the two blocks. **c)** Find the (common) acceleration of the two blocks and the tension in the string connecting the blocks.



Solution: The forces on m_1 are the weight $m_1 g$, tension T, and the normal force $F_{N,1}$; for m_2, the forces are the weight $m_2 g$, the tension T, and the normal force $F_{N,2}$. We use different (orthogonal) coordinate systems for the two blocks: along the incline and perpendicular to the incline. Along the incline: for the mass m_1,

$$T - m_1 g \sin 20^\circ = m_1 a,$$

and for m_2,

$$m_2 g \sin 50^\circ - T = m_2 a.$$

Perpendicular to the incline: for the mass m_1,

$$F_{N,1} - m_1 g \cos 20^\circ = 0,$$

and for the mass m_2,

$$F_{N,2} - m_2 g \cos 50^\circ = 0.$$

We express the tension T in terms of the acceleration a,

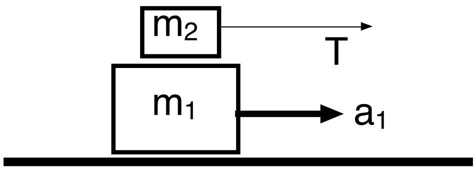
$$T = m_1 g \sin 20° + m_1 a.$$

so that $m_2 g \sin 50° - m_1 g \sin 20° - m_1 a = m_2 a$, and solve for a,

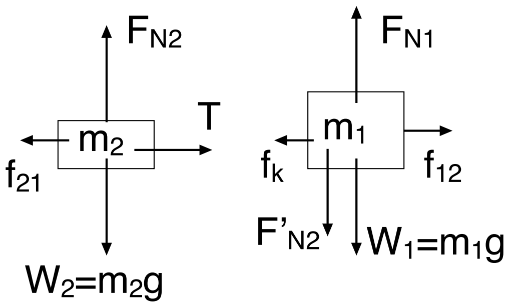
$$a = \frac{m_2 g \sin 50° - m_1 g \sin 20°}{m_1 + m_2} = \frac{4.5\,\mathrm{kg} \cdot \sin 50° - 2.5\,\mathrm{kg} \cdot \sin 20°}{2.5\,\mathrm{kg} + 4.5\,\mathrm{kg}} \cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} = 3.63\,\frac{\mathrm{m}}{\mathrm{s}^2}.$$

The tension follows

$$T = 2.5 \cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot \sin 20° + 2.5\,\mathrm{kg} \cdot 3.63\,\frac{\mathrm{m}}{\mathrm{s}^2} = 17.5\,\mathrm{N}.$$



Example 4: The block with mass $m_2 = 1.5\,\mathrm{kg}$ is pulled by the rope such that it slides across the block with mass $m_1 = 3.8\,\mathrm{kg}$. The block m_1 sits on a horizontal table. The coefficient of kinetic friction between the block m_1 and the table is $\mu_k = 0.17$. The coefficient of kinetic friction between the two blocks is unknown. The block m_1 moves towards the right with acceleration $a_1 = 0.56\,\mathrm{m/s}^2$. **a)** Draw the free-body diagram(s) appropriate for this problem. **b)** Write down Newton's second law that describes this system. **c)** What is the coefficient of kinetic friction between the blocks? **d)** The tension in the rope is $T = 11.2\,\mathrm{N}$. What is the acceleration a_2 of the block m_2?



Solution: The forces on the mass m_2 are the weight $m_2 g$ (down), the tension T (right), the normal force F_{N2} (up) exerted by the block m_1 on m_2, and the friction force between the blocks f_{21} (left). The forces on m_1 are the weight $m_1 g$ (down), the force $F'_{N,2}$ (down), the normal force $F_{N,1}$ extered by the table on m_1 (up), the friction force between the blocks f_{12} (right), and the kinetic friction force between m_1 and the table f_k (left). Note that we have two action-reaction pairs:

$$\vec{F}_{N,2} = -\vec{F}'_{N,2} \qquad \vec{f}_{12} = -\vec{f}_{21}.$$

The friction force f_k also has a reaction force: this is the force that the block m_1 exerts on the ground [that is, on the Earth]. We do not include the Earth in the mechanical system [since the Earth remains stationary], and can thus ignore this reaction force. We have for the mass m_2:

$$\sum F_x = T - f_{21} = m_2 a_2,$$

$$\sum F_y = F_{N2} - W_2 = 0,$$

with $f_{12} = \mu_k F_{N2}$, and for mass m_1,

$$\sum F_x = f_{12} - f_k = m_1 a_1,$$

$$\sum F_y = F_{N1} - W_1 - F'_{N2} = 0,$$

with $f_k = \mu_k F_{N1}$. We have $|f_{12}| = |f_{21}| = \mu_k m_2 g$ and $|f_k| = 0.17 \cdot (m_1 + m_2) g$. Thus, $\mu_k m_2 g - 0.17\ (m_1 + m_2) g = m_1 a_1$,

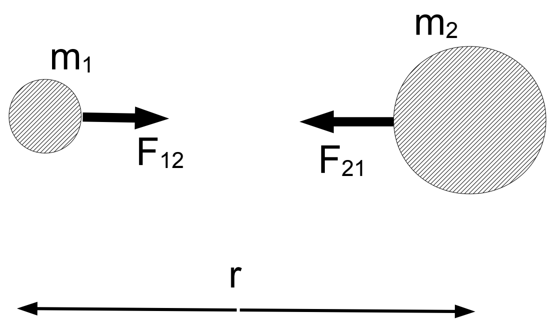
so that

$$\mu_k = \frac{m_1 a_1}{m_2 g} + 0.17 \frac{m_1 + m_2}{m_2} = \frac{3.8\,\mathrm{kg} \cdot 0.56\,\mathrm{m/s^2}}{1.5\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2}} + 0.17 \frac{1.5\,\mathrm{kg} + 3.8\,\mathrm{kg}}{1.5\,\mathrm{kg}} = 0.14 + 0.60 = 0.75.$$

We have $f_{21} = \mu_k m_2 g = 0.75 \cdot 1.5\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2} = 11.0\,\mathrm{N}$, and then

$$T - f_{12} = 11.2\,\mathrm{N} - 11.0\,\mathrm{N} = 0.2\,\mathrm{N} = 1.5\,\mathrm{kg} \cdot a_2 \quad \longrightarrow \quad a_2 = \frac{0.2\,\mathrm{N}}{1.5\,\mathrm{kg}} = 0.13\,\frac{\mathrm{m}}{\mathrm{s}^2}.$$

4.3 Gravitational Force



The weight $W = mg$ is the gravitational force that the Earth exerts on an object with mass m. In general, there is an attractive force between two masses m_1 and m_2 separated by the distance r.

We have two masses m_1 and m_2. The forces $F_{12} = -F_{21}$ are action-reaction pairs. The magnitude of the force is proportional to both masses, and inversely proportional to the square of the radius r,

$$F_{12} = F_{21} = G\frac{m_1 m_2}{r^2}, \tag{29}$$

where $G = 6.67 \times 10^{-11}\,\mathrm{N \cdot m^2/kg^2}$ is the *universal gravitational constant*.

We consider the case of mass m and the Earth M_E, so that $m_1 = m$ and $m_2 = M_E$. Since $r = R_E = 6.38 \times 10^6\,\mathrm{m}$ is the radius of the Earth, we get for the mass of the Earth: $mg = GM_E/R_E^2$:

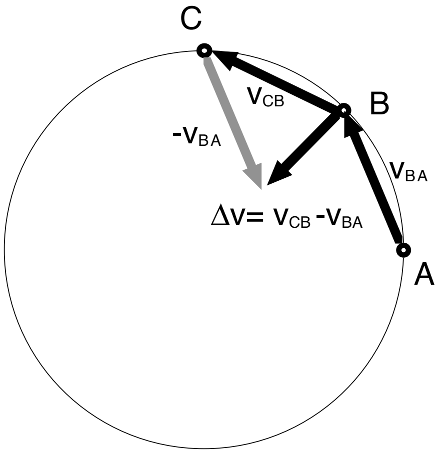
$$M_E = \frac{gR_E^2}{G} = \frac{9.8\,\mathrm{m/s^2} \cdot (6.38 \times 10^6\,\mathrm{m})^2}{6.67 \times 10^{-11}\,\mathrm{N^2\,m^2/kg^2}} = 5.98 \times 10^{24}\,\mathrm{kg}.$$

This agrees with the book value.

5 Uniform Circular Motion

In uniform circular motion, the object travels with constant speed v along a fixed circular trajectory with radius r. The speed and radius determine the period of the motion, i.e., the time to complete one full revolution:

$$v = \frac{\text{circumference}}{\text{period}} = \frac{2\pi r}{T} \quad \longrightarrow \quad T = \frac{2\pi r}{v}. \tag{30}$$



We gain insight by considering a numerical example. An object is moving on a circle with radius $R = 8.0\,\text{cm}$. The object is at point *A* at time $t = 0$, at point *B* at time $t = 2.0\,\text{s}$, and at point *C* at time $t = 4.0\,\text{s}$. The vectors representing the average velocities during the intervals $[0, 2\,\text{s}]$ and $[2\,\text{s}, 4\,\text{s}]$, $\vec{v}_{BA}$ and $\vec{v}_{CB}$, respectively, can be drawn, and the difference vector be found by geometric construction,

$$|\Delta\vec{r}| = |\vec{r}_B - \vec{r}_A| = |\vec{r}_C - \vec{r}_B| = 6.0\,\text{cm}.$$

The magnitudes of the velocities follows,

$$|\vec{v}_{AB}| = |\vec{v}_{BC}| = \frac{6.0\,\text{cm}}{2.0\,\text{s}} = 3.0\,\frac{\text{cm}}{\text{s}}.$$

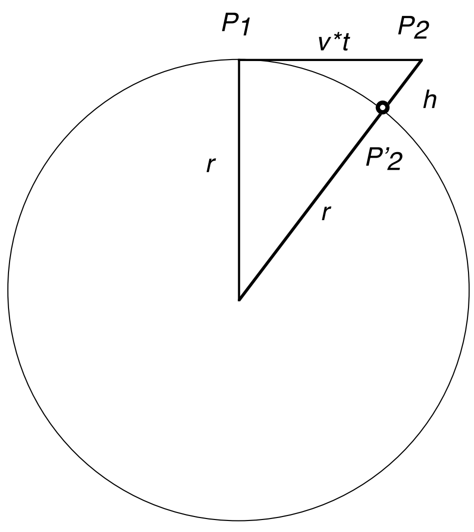
We then draw the vector $\Delta\vec{v} = \vec{v}_{CB} - \vec{v}_{BA}$, and find the magnitude,

$$|\Delta\vec{v}| = |\vec{v}_{CB} - \vec{v}_{BA}| = \frac{4.5\,\text{cm}}{2.0\,\text{s}} = 2.25\,\frac{\text{cm}}{\text{s}}.$$

The direction of $\vec{a}$ is in the direction of $\Delta\vec{v}$, i.e., towards the center,

$$|\vec{a}_{\text{ave}}(2.0\,\text{s})| = \frac{|\Delta\vec{v}|}{\Delta t} = \frac{\vec{v}(3.0\,\text{s}) - \vec{v}(1.0\,\text{s})}{3.0\,\text{s} - 1.0\,\text{s}} = \frac{2.25\,\text{cm/s}}{2.0\,\text{s}} = 1.1\,\frac{\text{cm}}{\text{s}^2}.$$

Thus, the acceleration is directed towards the center, so that the net force acting on the object must be directed towards the center as well.



We find an expression for the magnitude of the acceleration. At time $t = 0$, the object is at point P_1 moving with a velocity v in tangential direction. At time *t*, the object would be at point P_2. Instead, it moves along the circle, and thus is at point P_2'. We find an expression for the height using Pythagorean theorem: $r^2 + (vt)^2 = (r+h)^2$. We assume that the time interval is short, so that $vt < r$ and $h < r$. Since $(r+h)^2 = r^2 + 2rh + h^2 \simeq r^2 + 2rh$, we simplify the RHS: $r^2 + (vt)^2 \simeq r^2 + 2rh$ so that $h = (vt)^2/2r = \frac{1}{2}(v^2/r)t^2$. We compare the RHS with $x = at^2/2$, and find the value of the *centripetal acceleration*:

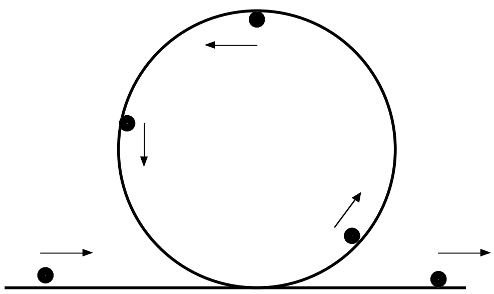
$$a_c = \frac{v^2}{r}. \tag{31}$$

For the numerical example above, we find $v^2/r \simeq (3\,\mathrm{cm/s})^2/(6\,\mathrm{cm}) = 1.5\,\mathrm{cm/s}^2$, in agreement with the graphical method.

We find the force acting towards the center from Newton's second law:

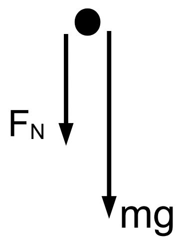
$$F_{\mathrm{net}} = ma_c = m\frac{v^2}{r}. \tag{32}$$

This is sometimes called "centripetal force;" however, there is no new force involved, and the centriptal force is the *net* force.



Example 1: An object goes around a vertical circular track, as shown. Find the minimum speed of the object near the top of the loop, so that it does not loose contact with the track.

Solution: We draw the free-body diagram of the object when it is near the top. We have the weight mg (down) and the normal force F_N (down).



Newton's second law then gives,

$$F_N + mg = m\frac{v^2}{r}.$$

The object barely completes the loop when it is 'weightless' near the top: $F_N = 0$. We thus find a condition for the smallest speed:

$$mg = m\frac{v_{\text{min}}^2}{r} \quad \longrightarrow \quad v_{\text{min}} = \sqrt{gr}.$$

For $r = 15.0\,\text{m}$, we find $v_{\text{min}} = \sqrt{9.8\,\text{m/s}^2 \cdot 15.0\,\text{m}} = 12.1\,\text{m/s}$.

Example 2: Calculate the length of a (sidereal) month from the distance between the Earth and the Moon $R_{EM} = 3.85 \times 10^8\,\text{m}$.

Solution: We find the mass of the Earth and the Moon from tables $M_E = 5.98 \times 10^{24}\,\text{kg}$ and $M_M = 7.35 \times 10^{22}\,\text{kg}$. We thus have $M_E >> M_M$, and assume that the Moon rotates about the center of the Earth:

$$F_{\text{net}} = G\frac{M_E M_M}{R_{EM}^2} = M_M \frac{v^2}{R_{EM}}.$$

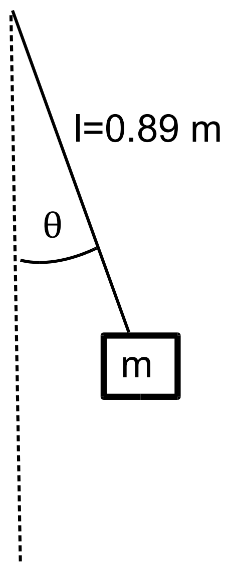
The orbit of the Moon is (nearly) circular, so that for the speed $v = 2\pi R_{EM}/T$, where T is the period [i.e., the length of a month], we find,

$$\frac{GM_E}{R_E^2} = \frac{4\pi^2 R_{EM}}{T^2} \quad\longrightarrow\quad T^2 = \frac{4\pi^2}{GM_E} R_{EM}^3.$$

We conclude that the square of the period is proportional to the third power of the radius of the orbit; this is essentially Kepler's third law of planetery orbits, for the special case when the orbit is a circle. We find the numerical value,

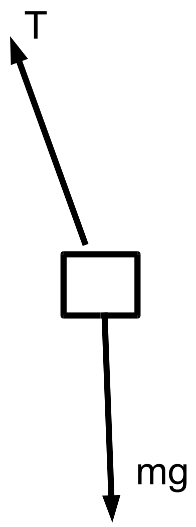
$$T = \sqrt{\frac{4\pi^2 (3.85 \times 10^8\,\mathrm{m})^3}{6.67 \times 10^{-11}\,\mathrm{N\,m^2/kg^2} \cdot 5.98 \times 10^{24}\,\mathrm{kg}}} = 2.38 \times 10^6\,\mathrm{s},$$

or 27.5 days (the value in tables is 27.3 days – the difference is due to simplifications made in our calculation).



Example 3: An object with mass $m = 1.6\,\mathrm{kg}$ is attached to a rope with length $l = 0.89\,\mathrm{m}$. It is brought in uniform circular motion, such that the rope makes an angle of $\theta = 16^\circ$ with the vertical. **a**) Find the period of the motion. **b**) Find the tension in the rope.

Solution: The forces acting on the block are the weight mg and the tension T.



We then have Newton's second law,

$$\sum F_x = -T\sin\theta = -ma_c = -m\frac{v^2}{r},$$

$$\sum F_y = T\cos\theta - mg = 0.$$

We find for the tension in the rope,

$$T = \frac{mg}{\cos\theta} = \frac{1.6\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}}{\cos 16^\circ} = 16.3\,\mathrm{N}.$$

We find from the horizontal component,

$$ma_c = T\sin\theta = mg\tan\theta = 1.6\,\mathrm{kg}\cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot\tan 16^\circ = 4.5\,\mathrm{N},$$

so that the centripetal acceleration is $a_c = 4.5\,\mathrm{N}/1.6\,\mathrm{kg} = 2.8\,\mathrm{m/s^2}$.

Now the radius of the circular trajectory is $r = l\sin\theta = 0.89\,\mathrm{m}\cdot\sin 16^\circ = 0.25\,\mathrm{m}$. The velocity follows,

$$v = \sqrt{a_c r} = \sqrt{2.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 0.25\,\mathrm{m}} = 0.84\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The period then follows $T = 2\pi r/v = 2\pi\cdot 0.25\,\mathrm{m}/(0.84\,\mathrm{m/s}) = 1.9\,\mathrm{s}$.

6 Work and Energy

When a person jumps off the ground from standing, she crouches and pushes on the ground with a force greater than weight. As a result, the ground exerts a normal force ["ground reaction force"] greater than the weight of the person: $F_N - mg = F_{\text{net}} > 0$ (upwards). The person could remain in this (uncomfortable) position forever; a non-zero net force is not sufficient to generate a finite velocity at lift-off. She can accomplish this by stretching her legs. If c is the crouching distance, we say that the net force is doing work $W = F_{\text{net}} \cdot c$. This work generates the necessary finite velocity at lift-off.

6.1 Work-Kinetic Energy

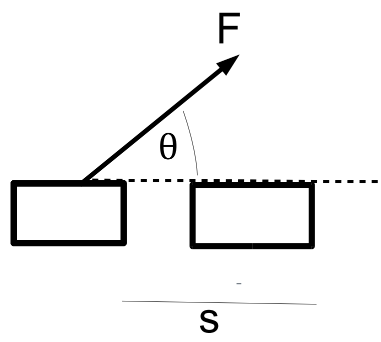
We start from the kinematics equation in one dimension with time eliminated, $v^2 - v_0^2 = 2a\Delta x$. We multiply by the mass of the object and re-arrange,

$$\frac{m}{2}v^2 - \frac{m}{2}v_0^2 = ma \cdot \Delta x = F \cdot \Delta x. \tag{33}$$

We define the *kinetic energy*

$$\text{KE} = \frac{m}{2}v^2. \tag{34}$$

We note that the kinetic energy depends on the magnitude of the velocity, but not the direction of motion; the LHS of Eq. (34) is the change in kinetic energy.



For the RHS we define the *work* as,

$$W = Fs\cos\theta. \tag{35}$$

We then find the *work-kinetic energy theorem*:

$$W = \Delta\text{KE} = \text{KE} - \text{KE}_0. \tag{36}$$

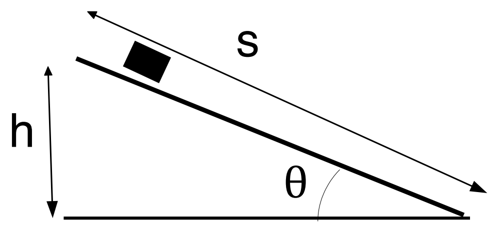
The units of work [and the kinetic energy] are: $[W] = [\text{KE}] = [F] \cdot [s] = \text{N} \cdot \text{m} = \text{J}$, or Joules.

We consider a few illustrative cases: (1) if a ball is in free fall, the weight mg is the only force acting on the ball. If the ball travels upwards, the work done by the weight is negative, such that the kinetic energy decreases, $\Delta\text{KE} < 0$, and the ball slows down. Likewise, if the ball falls downwards, the work done by the weight is positive and the change in kinetic energy is positive, $\Delta\text{KE} < 0$, and the ball speeds up. (2) When an object slides across a frictionless horizontal surface [e.g., an ice skater], the forces acting on the skater are the weight and the normal force that are both in a vertical direction. Thus, we have for the angle $\theta = 90°$, so that $\cos\theta = 0$ and the work done by both forces is zero, $W = 0$. This implies that the kinetic energy of the object does not change $\Delta\text{KE} < 0$, and the speed of the object remains constant.

The normal force is always perpendicular to the surface, while displacement is parallel to the surface. Thus, the angle θ between the normal force and the displacement is $\theta = 90°$, and the work is equal to zero $W = 0$. The normal force is an example of a contstraint force: work done by constraint forces are always zero.

When a person holds a weight and stretches the arms, the arm muscles i obviously do work [felt by *fatiguing* of the muscles], and yet the mechanical work defined in Eq. (35) is zero. This example shows that our intuitive sense of work can sometimes be different from the definition used in physics. The cells in the muscles produce tension by continuously stretching and contracting. Thus, microscopic cell displacements are parallel to the force (tension), so muscle cells do non-zero work.

If a racket hits a tennis ball, the ball will accelerate so that it acquires the speed v starting from rest v_0. The force exerted by the racket on the ball is $W = mv^2/2$, where m is the mass of the ball. If the tennis ball is distorted by a distance s [much smaller than the radius of the ball], the average force is $F_{\text{ave}} = W/s$. We see that the force becomes larger $F_{\text{ave}} \to \infty$ as the distortion becomes smaller $s \to 0$. This shows that there is no such thing as a completely rigid object; even a golf or steel ball is distorted slightly when hit by another object.



Example 1: An object with mass $m = 3.2\,\text{kg}$ sits on an incline plane at an angle $\theta = 23°$. The object slides down the incline with $s = 1.9\,\text{m}$, starting from rest. Find the speed of the ball at the end of the ramp.

Solution: The net force on the object is equal to the component of weight parallel to the incline, $F_{\text{net}} = -mg\sin\theta$; here the negative sign implies that the force is directed downward. We thus get for the work:

$$W = F_{\text{net}} \cdot s = mg\sin\theta \cdot s = mgh,$$

where we use $s\sin\theta = 1.9\,\text{m}\sin 23° = 0.74\,\text{m} = h$ for the height of the incline. The work-kinetic energy theorem follows,

$$\frac{m}{2}v^2 = mgh \quad \longrightarrow \quad v = \sqrt{2gh} = \sqrt{2 \cdot 9.8\,\frac{\text{m}}{\text{s}^2} \cdot 0.74\,\text{m}} = 3.8\,\frac{\text{m}}{\text{s}}.$$

This result is remarkable in two ways: (1) The speed is independent of the mass, so that all objects will reach the bottom at the end of the ramp. (2) The final speed is independent of the angle θ: it is the same for an object sliding along the incline or freely falling object with $\theta = 90°$. While the final speed is independent of the angle θ, the time to reach the bottom of the ramp depends on the incline θ. We find the time from the kinematics equation for motion with constant acceleration $a = g \sin\theta$.

6.2 Potential Energy

The result shows that work done by gravitational force [weight] is the same, whether the object slides down along the incline or falls down the horizontal and then travels along the horizontal. That is, the work is independent of the path: we say the gravitational force is *conservative*.

It follows that the work done by a conservative force can be expressed as a difference of a *potential energy* (PE),

$$W = -\left[\mathrm{PE}_f - \mathrm{PE}_0\right]. \tag{37}$$

The potential energy is a function of the coordinates. Since $h = |\Delta y|$, the gravitational potential energy is given by

$$\mathrm{PE} = mgy. \tag{38}$$

Note that the choice of coordinates $y = 0$ is arbitrary, so that the value of the potential energy is also arbitrary. We insert Eq. (37) into Eq. (36) and find $\mathrm{KE} - \mathrm{KE}_0 = -[\mathrm{PE} - \mathrm{PE}_0]$ or,

$$\mathrm{KE} + \mathrm{PE} = \mathrm{KE}_0 + \mathrm{PE}_0. \tag{39}$$

We define the sum of kinetic and potential energy as the total mechanical energy, and find

$$E_{\mathrm{mech}} = \mathrm{KE} + \mathrm{PE} = \mathrm{const}. \tag{40}$$

Equation (40) then represents the *conservation of (mechanical) energy*.

Example 2: A steelball with mass $m = 0.44\,\mathrm{kg}$ and radius $r = 1.0\,\mathrm{cm}$ moves along a rollercoaster. Seven photogates are placed along the track at different heights h_i from the base of the track; we measure the time intervals Δt_i for the steel ball to pass through each photogate,

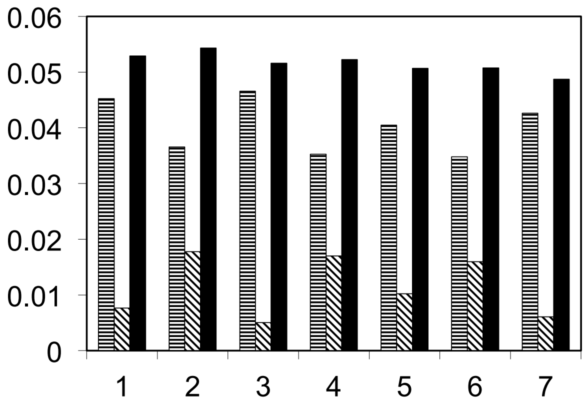
Photogate	h_i [m]	Δt_i [s]
1	0.104	0.0420
2	0.084	0.0275
3	0.107	0.0516
4	0.081	0.0281
5	0.093	0.0363
6	0.080	0.0291
7	0.098	0.0426

Calculate the gravitational potential energy, the kinetic energy, and the total mechanical energy.

Solution: We have the gravitational potential energy $\mathrm{PE}_i = mgh_i$. The (average) speed of the ball is $v_i = 2r/\Delta t_i$ as it passes through each photogate. The kinetic energy then follows $\mathrm{KE}_i = mv_i^2/2$. Note that the steel ball moves along the track (translational motion), but also rotates [rotational motion]. As a result, the kinetic energy of the ball has a contribution from rotation. In this situation, the contribution of rotational kinetic energy can be described by an "effective" mass m^*, greater than the mass m. If the track has a width $w = 0.95\,\mathrm{cm}$, we find,

$$m \longrightarrow m^* = \left[1 + \frac{2}{5}\frac{r^2}{r^2 - (w/2)^2}\right] m = 1.517\, m.$$

The (total) kinetic energy is given by $\mathrm{KE}_i = m^* v_i^2/2$,



Photogate	KE_i [J]	PE_i [J]	$\mathrm{E}_{\mathrm{mech},i}$ [J]
1	0.0453	0.0076	0.0529
2	0.0366	0.0178	0.0543
3	0.0466	0.0051	0.0516
4	0.0353	0.0170	0.0523
5	0.0405	0.1002	0.0507
6	0.0348	0.0160	0.0508
7	0.0426	0.0061	0.0487

Discussion: The mechanical energy of the system (slightly) decreases due to friction.

Example 3: We return to the problem of the vertical loop from Chapter 4, where we found that the minimum speed near the top is given by $v_{\min} = \sqrt{gR}$, with R as the radius of the loop.

Solution: We assume that the object is released from rest, so that the initial mechanical energy is equal to the potential energy. We choose $y = 0$ at the bottom of the loop. Then

$$mgH = \frac{m}{2}gR + mg2R = \frac{5}{2}mgR \quad \longrightarrow \quad H = \frac{5}{2}R.$$

Discussion: The rolling ball has to be released from a greater height than an object sliding down a frictionless track.

6.3 Non-conservative forces

For an object in the gravitational field of the Earth, the mechanical system is the object and the Earth. The force [interaction] is described by a potential energy energy: we say that the system is *closed*. Not all interactions can be described in this manner. Since the (kinetic) friction force is always opposing the displacement, the work done kinetic friction force is negative. In particular, the work is non-zero when the object returns to its starting position, and we conclude that (kinetic) friction is a non-conservative force. Systems for which interactions are described by non-conservative forces are called *open systems*.

We notice that the mechanical energy in Example 2 decreases from photogate 1 to 7; this decrease stems from the work done by the friction force. We write the work done by the friction force as,

$$W_{\mathrm{nc}} = -f_k s < 0, \tag{41}$$

where f_k is the (average) kinetic friction force and s is the distance along the track. The change in the mechanical energy can be written,

$$\Delta E_{\mathrm{mech}} = E_{\mathrm{mech},f} - E_{\mathrm{mech},i} = W_{\mathrm{nc}}, \tag{42}$$

and always decreases, $\Delta E_{\mathrm{mech}} < 0$.

Example 4: An object with mass $m = 3.2\,\mathrm{kg}$ sits on an incline plane at an angle $\theta = 23°$. The object slides down the incline with $s = 1.9\,\mathrm{m}$, starting from the rest and travelling at the speed $v = 2.1\,\mathrm{m/s}$ at the end of the ramp. Find the coefficient of kinetic frcition between the object and the ramp [cf. Example 1].

Solution: We find the mechanical energy when the object is on the top of the ramp and at the bottom,

$$E_{\mathrm{mech},i} = mgh = 3.2\,\mathrm{kg} \cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 0.74\,\mathrm{m} = 23.2\,\mathrm{J},$$
$$E_{\mathrm{mech},f} = \frac{m}{2}v_f^2 = \frac{3.2\,\mathrm{kg}}{2}\left(2.1\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 7.1\,\mathrm{J}.$$

We thus find the work done by the friction force,

$$W_{\mathrm{nc}} = 7.1\,\mathrm{J} - 23.2\,\mathrm{J} = -16.1\,\mathrm{J},$$

so that for the friction force

$$f_k = -\frac{W_{\mathrm{nc}}}{s} = -\frac{-16.1\ \mathrm{J}}{1.9\,\mathrm{m}} = 8.5\,\mathrm{N}.$$

The normal force is given by $F_N = mg\cos\theta = 3.2\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s}^2 \cos 23° = 28.9\,\mathrm{N}$. Since $f_k = \mu_k F_N$, we find the coefficient of kinetic friction,

$$\mu_k = \frac{f_k}{F_N} = \frac{8.5\,\mathrm{N}}{28.9\,\mathrm{N}} = 0.29.$$

This is a reasonable value for the coefficient of kinetic friction.

7 Momentum

When a ball bounces off a wall, it is reflected with the same speed but travels in the opposite direction: $\vec{v} \to -\vec{v}$. The kinetic energy is constant, and the net work done on the ball by the wall is zero. Since $\Delta\vec{v} = \vec{v}_f - \vec{v}_i = -2\vec{v} = \vec{a}_{\text{ave}}\Delta t = \vec{F}\Delta t/m$, we define the *impulse* as $\text{force} \times \text{time}$,

$$\vec{J} = \vec{F}\,\Delta t. \tag{43}$$

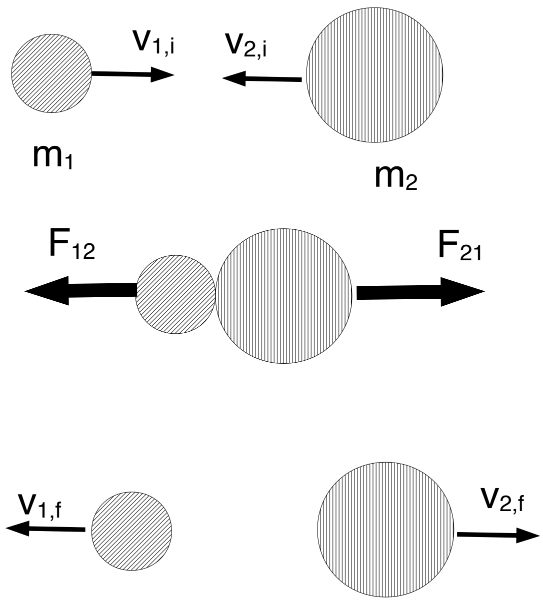
So the momentum of an object with mass m as $\text{mass} \times \text{velocity}$ is

$$\vec{p} = m\vec{v}. \tag{44}$$

Newton's second law, $\vec{F} = m\vec{a}$, can then be written,

$$\vec{J} = \Delta\vec{p}. \tag{45}$$

This relationship is somewhat analogous to the work-kinetic energy theorem. It allows us to describe changes in the velocity of an object without knowing details of the forces [the magnitudes and direction an dfor how long they last]. For the ball bouncing off the wall, we do not know the magnitude of the force exerted by the wall on the ball or the duration of the force; however, the impulse of the force has a known value, $\vec{J} = -2m\vec{v}$. The unit of momentum and impulse are $[J] = [p] = \mathrm{N} \cdot \mathrm{s} = \mathrm{kg} \cdot \mathrm{m/s}$.



We have two objects [pucks] sliding on (idealized) ice and traveling at velocities $\vec{v}_{1,i}$ and $\vec{v}_{2,i}$, respectively. They collide and exert forces $\vec{F}_{12}$ and $\vec{F}_{21}$ on each other. These two forces are an action-reaction pair so that $\vec{F}_{12} = -\vec{F}_{21}$. After the collision, they travel at velocities $\vec{v}_{1f}$ and $\vec{v}_{2f}$. Note that there no external forces, for example the gravitational force in the case when the surface is inclined.

We have for the impulses,

$$\begin{aligned} F_{12}\Delta t &= m\Delta v_1 = m_1(v_{1,f} - v_{1,i}), \\ F_{21}\Delta t &= m\Delta v_2 = m_2(v_{2,f} - v_{2,i}). \end{aligned}$$

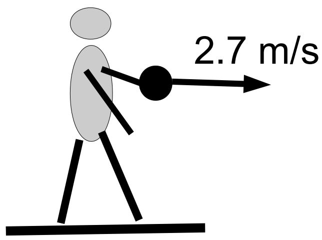
Newton's third law states $\vec{F}_{12} = -\vec{F}_{21}$, or $F_{12} + F_{21} = 0$, so that the addition of the two equations yields $(F_{12} + F_{21})\Delta t = m_1(v_{1,f} - v_{1,i}) + m_2(v_{2,f} - v_{2,i}) = 0$ or,

$$m_1 v_{1,i} + m_2 v_{2,i} = m_1 v_{1,f} + m_2 v_{2,f}. \tag{46}$$

We define the total momentum of the system [mass m_1 and mass m_2],

$$p_{\text{tot}} = m_1 v_1 + m_2 v_2. \tag{47}$$

We thus see that the total momentum is the same before and after the collision. We say that the total momentum is conserved. No detailed knowledge of the forces is necessary.



Example 1: A person [mass $m_p = 43\,\mathrm{kg}$] is standing on a skateboard [mass $m_b = 2\,\mathrm{kg}$] at rest, while holding a dumbbell [mass $M = 5\,\mathrm{kg}$]. The person throws the dumbbell with a speed of 2.7 m/s to the right.

Solution: System 1 is the person and skateboard $m_1 = 45\,\mathrm{kg}$ and system 2 is the dumbbell is $m_2 = 5\,\mathrm{kg}$. Initially, both systems are at rest $v_{1,i} = v_{2,i} = 0$, so that the total momentum is zero,

$$p_{\text{tot}} = 0.$$

There are no external forces, so the total momentum is conserved such that the total momentum remains zero after the throw ["collision"]. Since $v_{2.f} = 2.7\,\mathrm{m/s}$, we find,

$$p_{\text{tot}} = 0 = 45\,\mathrm{kg} \cdot v_{1,f} + 5\,\mathrm{kg} \cdot 2.7\,\frac{\mathrm{m}}{\mathrm{s}} \qquad \longrightarrow \qquad v_{1,f} = -\frac{5\,\mathrm{kg} \cdot 2.7\,\mathrm{m/s}}{45\,\mathrm{kg}} = -0.3\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The person (and skateboard) are moving towards the 'left.' The force exerted by the person on the dumbbell is directed towards the 'right,' and the force exterted by the dumbbell on the person is directed towards the 'left.' We say that the person (and skateboard) 'recoil.'

Discussion: The total momentum before and after the collision is zero. While the initial kinetic energy is zero $\mathrm{KE}_i = 0$, the final kinetic energy is non-zero,

$$\mathrm{KE}_f = \frac{45\,\mathrm{kg}}{2}\left(0.3\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \frac{5\,\mathrm{kg}}{2}\left(2.7\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 2.0\,\mathrm{J} + 18.2\,\mathrm{J} = 20.2\,\mathrm{J}.$$

Since $W = \mathrm{KE}_f - \mathrm{KE}_i = 20.2\,\mathrm{J}$, non-zero work is being done on the entire system [i.e., person and skateboard *plus* dumbbell]. This work is done by the person's muscles [they are not considered part of the mechanical system, and are thus "external"]. The same mechanism applies to the locomotion of squids [with the squid replacing the person (and skateboard), and water replacing the dumbbell]. The propulsion of rockets and space probes [Juno, Huygens, Voyager, etc.] is based on the conservation of momentum.

7.1 Center of Mass

The conservation of momentum can be expressed in a different way. We write the velocities in terms of displacements $v = \Delta x/\Delta t = (x - x_0)/\Delta t$, so that

$$\begin{aligned} p_{\rm tot} &= m_1\frac{x_1 - x_{1,0}}{\Delta t} + m_2\frac{x_2 - x_{2,0}}{\Delta t} \\ &= \frac{1}{\Delta t}\left[(m_1x_1 + m_2x_2) - (m_1x_{1,0} + m_2x_{2,0})\right]. \end{aligned} \tag{48}$$

We define the center of mass [CoM] in one-dimension,

$$x_{\rm CoM} = \frac{m_1x_1 + m_2x_2}{m_1 + m_2}. \tag{49}$$

Note that the CoM is a mathematical construct and is not associated with a particular mass. The (fictional) point CoM moves at the velocity

$$v_{\rm CoM} = \frac{\Delta x_{\rm CoM}}{\Delta t}. \tag{50}$$

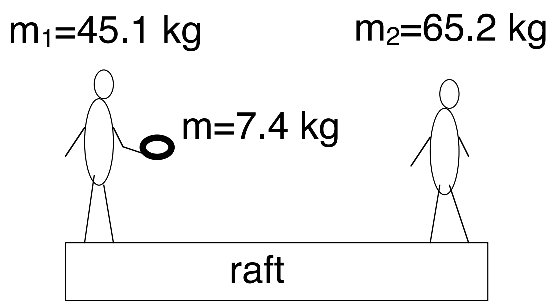
We see that the RHS of Eq. (48) is proportional to $v_{\rm CoM}$. We find

$$p_{\rm tot} = Mv_{\rm CoM}, \tag{51}$$

where $M = m_1 + m_2$ is the total mass of the entire system. It follows,

$$F_{\rm ext} = \frac{\Delta p_{\rm tot}}{\Delta t}. \tag{52}$$

Eqs. (49)-(52) are valid for more than two objects with the obvious generalization. Thus, a multiparticle system moves under the influence of an external force $\vec{F}_{\rm ext}$, as if the entire mass is concentrated in the CoM.



Example 2: Two people with masses $m_1 = 45.1\,\mathrm{kg}$ and $m_2 = 65.2\,\mathrm{kg}$ are standing on the left and right side of a $4.2\,\mathrm{m}$ -long massless raft, as shown. The person on the left holds a ball with mass $m_b = 7.4\,\mathrm{kg}$. **a)** Find the center of mass of the system. **b)** The person with mass m_1 throws the ball to the person with mass m_2. Find the displacement of the raft after the second person has caught the ball.

Solution: We have for the total mass $M_{\mathrm{tot}} = m_1 + m_2 + m_b = 45.1\,\mathrm{kg} + 65.2\,\mathrm{kg} + 7.4\,\mathrm{kg} = 117.7\,\mathrm{kg}$. We choose $x = 0$ at the left side of the boat,

$$M_{\mathrm{tot}} x_{\mathrm{CoM}} = (45.1\,\mathrm{kg} + 7.4\,\mathrm{kg}) \cdot 0 + 65.2\,\mathrm{kg} \cdot 4.2\,\mathrm{m} = 273.8\,\mathrm{kg} \cdot \mathrm{m},$$

so that for the center of mass,

$$x_{\mathrm{CM}} = \frac{273.8\,\mathrm{kg} \cdot \mathrm{m}}{117.7\,\mathrm{kg}} = 2.33\,\mathrm{m}.$$

We use Δx for the displacement of the boat. Then

$$M_{\mathrm{tot}} x_{\mathrm{CM}} = m_1 \cdot \Delta x + (m_2 + m_b) \cdot (4.2\,\mathrm{m} + \Delta x) = M_{\mathrm{tot}} \cdot \Delta x + 304.9\,\mathrm{kg} \cdot \mathrm{m}.$$

We thus have $273.8\,\mathrm{kg}\cdot\mathrm{m} = 117.7\,\mathrm{kg}\cdot\Delta x + 304.9\,\mathrm{kg}\cdot\mathrm{m}$, or

$$\Delta x = \frac{273.8\,\mathrm{kg}\cdot\mathrm{m} - 304.9\,\mathrm{kg}\cdot\mathrm{m}}{117.7\,\mathrm{kg}} = -0.26\,\mathrm{m},$$

or $\Delta x = -26\,\mathrm{cm}$. That is, the raft is moving towards the *left*.

Discussion: The person m_1 exerts a force on the ball directed towards the *right*. The ball exerts a reaction force on the person towards the *left*. Note that the conservation of momentum does not tell us the duration of the collision, i.e., the time for the boat to move to its new position.

7.2 Collisions

The momentum is conserved, but the mechanical energy is *not conserved* in the example with a person throwing the dumbbell while standing on a skateboard. This type of interaction between objects is called *inelastic collision*.

Example 3: A bullet [with mass $m = 23\,\mathrm{g}$ and velocity $v_b = 304\,\mathrm{m/s}$] is fired into a block [with mass $M = 3.1\,\mathrm{kg}$] that is initially at rest. The bullet embeds itself $6.2\,\mathrm{cm}$ into the block. **a)** What is the speed of the block after the bullet embeds itself? **b)** What is the average force exerted by the block on the bullet?

Solution: This is an *inelastic* collision, so that for the speed *V*,

$$mv_b = (M+m)V \quad \longrightarrow \quad V = \frac{0.023\,\mathrm{kg}}{3.1\,\mathrm{kg} + 0.023\,\mathrm{kg}} \cdot 304\,\frac{\mathrm{m}}{\mathrm{s}} = 2.24\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We use the work-kinetic energy theorem for the bullet,

$$W = \Delta\mathrm{KE} = \frac{0.023\,\mathrm{kg}}{2}\left[\left(2.24\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - \left(304\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2\right] = -1063\,\mathrm{J}.$$

Since the force acts over the distance $s = 0.062\,\mathrm{m}$, we get

$$W = -f_{\mathrm{ave}}s \qquad \longrightarrow \qquad f_{\mathrm{ave}} = \frac{1063\,\mathrm{J}}{6.2\times10^{-2}\,\mathrm{m}} = 1.7\times10^4\,\mathrm{N}.$$

When both momentum and mechanical energy are conserved, the collision is said to be *completely elastic*,

$$m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}, \tag{53}$$

$$\frac{m_1}{2}v_{1i}^2 + \frac{m_2}{2}v_{2i}^2 = \frac{m_1}{2}v_{1f}^2 + \frac{m_2}{2}v_{2f}^2. \tag{54}$$

We thus have two equations for two unknowns v_{1f} and v_{2f}, assuming that the masses of the two colliding objects are known, $(v_{1i}, v_{2i}) \quad\longrightarrow\quad (v_{1f}, v_{2f})$.

Example 4: We measure the collision of two gliders on a (frictionless) air track, when one of the gliders is initially at rest:

Case	m_1[kg]	m_2 [kg]	v_{1i} [m/s]	v_{2i} [m/s]	v_{1f} [m/s]	v_{2f} [m/s]
1	0.45	0.46	0.64	0	0	0.59
2	0.45	0.65	0.63	0	-0.14	0.47
3	0.65	0.45	1.01	0	0.14	1.16

a) Calculate the total momentum and total energy before and after the collision. Are the conservation laws obeyed? **b)** Is Newton's third law obeyed during the collision? **c)** Describe the collision for an observer at the center-of-mass [i.e., a moving reference frame rather than a fixed laboratory system].

Solution: We find

Case	$P_{i,\text{tot}}$ [kg · ms]	$P_{f,\text{tot}}$ [kg · ms]	$E_{i,\text{tot}}$ [J]	$E_{f,\text{tot}}$ [J]
1	0.27	0.27	0.09	0.08
2	0.28	0.24	0.09	0.08
3	0.65	0.61	0.33	0.31

We observe that $\vec{P}_i = \vec{P}_f$ and $E_{i,\text{tot}} = E_{f,\text{tot}}$; momentum and energy are conserved. Since $\vec{F}\Delta t = \Delta p$, we calculate the change of the momentum or impulse for each mass, $\Delta p = p_f - p_i = m(v_f - v_i)$. We get

Case	Δ p_1 [kg · m/s]	Δ p_2 [kg · m/s]
1	-0.29	+0.27
2	-0.35	+0.31
3	- 0.57	+0.52

We conclude that $\Delta p_1 = -\Delta p_2$, which is in agreement with Newton's third law.

The CoM velocity follows from the total momentum $v_{\text{CoM}} = P/M$, where $M = m_1 + m_2$ is the total mass. The *relative* velocity and *relative* momentum are defined,

$$\bar{v}_i = v_i - v_{\text{CoM}}, \quad \bar{p}_i = m_i \bar{v}_i.$$

We get

Case	$\bar{p}_{1,i}$ [kg · m/s]	$\bar{p}_{1,f}$ [kg · m/s]	$\bar{p}_{2,i}$ [kg · m/s]	$\bar{p}_{2,f}$ [kg · m/s]
1	+0.15	-0.13	-0.15	+0.13
2	+0.17	-0.16	-0.17	+0.16
3	+0.27	-0.27	-0.27	+ 0.27

We observe that the relative momenta are exchanged during the collision,

$$\bar{p}_{1,f} = \bar{p}_{2,i} \qquad \bar{p}_{2,f} = \bar{p}_{1,i}.$$

The collision can be described in terms of center of mass and relative momentum. For $v_{2,i} = 0$ we have

$$v_{\text{CoM}} = \frac{m_1}{m_1 + m_2} v_{1,i}. \tag{55}$$

We then find,

$$\bar{p}_{1i} = -\bar{p}_{2i} = \frac{m_1 m_2}{m_1 + m_2} v_{1,i}. \tag{56}$$

We find the final velocities $v_{1f} = v_{\text{CoM}} - \bar{p}_{1i}/m_1$ and $v_{2,f} = v_{\text{CoM}} - \bar{p}_{2i}/m_2$, or

$$v_{1,f} = \frac{m_1}{m_1 + m_2} v_{1i} - \frac{m_2}{m_1 + m_2} v_{1i} = \frac{m_1 - m_2}{m_1 + m_2} v_{1,i}, \tag{57}$$

$$v_{2,f} = \frac{m_1}{m_1 + m_2} v_{1i} - \left(-\frac{m_1}{m_1 + m_2} v_{1i} \right) = \frac{2m_1}{m_1 + m_2} v_{1,i}. \tag{58}$$

It is worthwhile to consider special cases: (1) $m_1 = m_2$ so that $v_{1,f} = 0$ and $v_{2,f} = v_{1,f}$; the moving object comes to a full stop, and the object initially at rest begins to move. This can be observed when a penny is "flicked" and slides towards a penny at rest. (2) $m_1 << m_2$ so that $v_{1,f} = -v_{1,i}$ and $v_{2,f} \simeq 0$; this describes the situation in which a ball hits a wall [*anchored to the Earth*], with the ball bouncing off with the same speed but in the opposite direction, such that the wall [Earth] does not move. (3) $m_1 >> m_2$ so that $v_{1,f} \simeq v_{1,i}$ and $v_{2,f} \simeq 2v_{1,i}$; the heavier object is barely affected by the collision, while the lighter object moves with twice the speed.

8 Rotational Kinematics and Dynamics

Rotational kinematics and dynamics can be described similarly to linear kinematics and dynamics, provided that appropriate quantities are used.

8.1 Rotational Kinematics

Angular displacement is defined,

$$\Delta\theta = \frac{\text{arc length}}{\text{radius}} = \frac{s}{r}. \tag{59}$$

For a full circle $\theta = 2\pi\,\mathrm{rad}$. Note that rad is not a unit, but rather a placeholder. Thus, "rad" may appear and disappear from equations. The angular velocity is defined

$$\omega = \frac{\Delta\theta}{\Delta t} \qquad [\omega] = \frac{\text{rad}}{\text{s}}, \tag{60}$$

and angular acceleration,

$$\alpha = \frac{\Delta\omega}{\Delta t}, \qquad [\alpha] = \frac{\text{rad}}{\text{s}^2}. \tag{61}$$

If T is the *period*, i.e., the time for one revolution is the average speed and is given by,

$$\omega_{\text{ave}} = \frac{2\pi\,\text{rad}}{T}. \tag{62}$$

Example 1: Find the angular speed of the minute and second hand.

Solution: We have the period for the second hand $T_{\text{second}} = 60\,\text{s}$, so that

$$\omega_{\text{second}} = \frac{2\pi\,\text{rad}}{60\,\text{s}} = 0.105\,\frac{\text{rad}}{\text{s}}.$$

The period for the minute hand is $T = 3600\,\text{s}$, so that

$$\omega_{\text{minute}} = \frac{2\pi\,\text{rad}}{3600\,\text{s}} = 1.75\times10^{-3}\,\frac{\text{rad}}{\text{s}}.$$

Discussion: Another unit for angular speed in common use is *revolutions per minute* (rpm). Then $\omega_{\text{second}} = 1\,\text{rpm}$ and $\omega_{\text{minute}} = 60^{-1}\,\text{rpm} = 0.0167\,\text{rpm}$.

The kinematics equations are derived the same way as in the case of linear equations. For $\alpha = \text{const}$, we find the average velocity $\omega_{\text{ave}} = (\omega_0 + \omega)/2 = (\omega_0 + \alpha t/2)$, so that for the angular displacement we have $\Delta\theta = \omega_{\text{ave}} t = \omega_0 t + \alpha t^2/2$. We eliminate the time from the equation for the angular velocity $t = (\omega - \omega_0)/\alpha$, so that $\Delta\theta = \omega_{\text{ave}} t = (\omega_0+\omega)/2\cdot(\omega-\omega_0)/\alpha = (\omega^2 - \omega_0^2)/2\alpha$. We thus arrive at the kinematics equation for rotation,

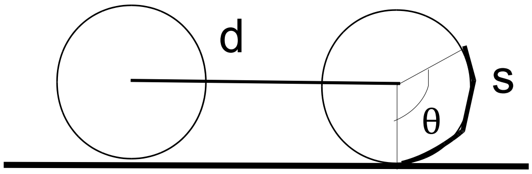
$$\omega(t) \;=\; \omega_0 + \alpha t, \tag{63}$$

$$\theta(t) \;=\; \theta_0 + \omega_{\text{ave}} t, \quad \text{with} \;\; \omega_{\text{ave}} = [\omega + \omega_0]/2, \tag{64}$$

$$\theta(t) \;=\; \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2, \tag{65}$$

$$\omega^2 \;=\; \omega_0^2 + 2\alpha\Delta\theta. \tag{66}$$

We note that these equations are identical to Eqs. (11)-(15), when quantities for linear motion (x, v, a) are replaced by the respective quantities for rotational motion (θ, ω, α).



Rolling Motion: We consider a wheel with radius r that rolls *without slipping*. This means that the distance *d* is equal to arc-length $s = r\theta$,

$$d = r\theta. \tag{67}$$

If the center of the wheel travels the distance *d* in the time *t*, we have for the linear speed $v = d/t$, and for the angular speed $\omega = \theta/t$. We find $v = (r\theta)/t = r(\theta/t)$ or

$$v_t = r\omega. \tag{68}$$

Similarly, if the center of the wheel accelerates from $v_0 = 0$ to the velocity v in time *t*, the angular velocity increases from ω_0 to ω . Since $v = r\omega$, we find

$$a_t = r\alpha. \tag{69}$$

Here, v and a are the velocity and acceleration of a point on the perimeter relative to the the center: i.e., they are the *tangential* velocity and acceleration, $v_T = \omega r$ and $a_T = \alpha r$.

Example 2: **a)** A car travels at the speed $v = 25\,\mathrm{m/s}$. Find the angular speed of the wheels when the radius of the wheel is $r = 0.31\,\mathrm{m}$ (1 foot). **b)** If the car accelerates from rest to $v = 25\,\mathrm{m/s}$ in the time $t = 17\mathrm{s}$, find the angular acceleration of the wheel, and the number of turns of the wheel.

Solution: We find for the angular speed,

$$\omega = \frac{v}{r} = \frac{25\,\mathrm{m}}{\mathrm{s}} = 80.6\,\frac{\mathrm{rad}}{\mathrm{s}}.$$

We find for the angular acceleration,

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{80.6\,\mathrm{rad/s} - 0}{17\,\mathrm{s}} = 4.74\,\frac{\mathrm{rad}}{\mathrm{s}^2}.$$

We have $\omega^2 = 2\alpha\Delta\theta$, so that

$$\Delta\theta = \frac{\omega^2}{2\alpha} = \frac{(80.6\,\mathrm{rad/s})^2}{2\cdot 4.74\,\mathrm{rad/s}^2} = 685.3\,\mathrm{rad}.$$

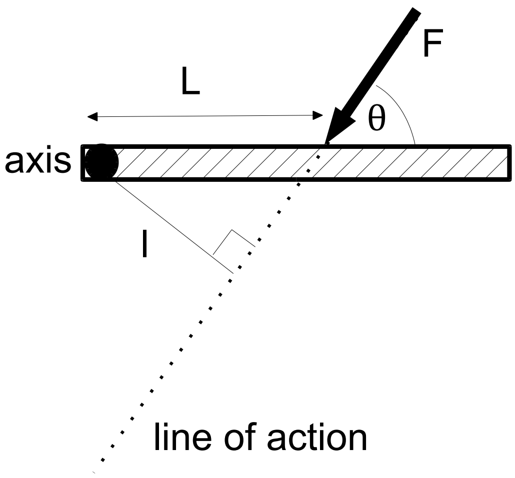
Since $1\,\mathrm{rev} = 2\pi\,\mathrm{rad}$, the number of turns follows $N = \Delta\theta/(2\pi\,\mathrm{rad}) = 109.1$.

The speed is constant for uniform circular motion speed. Since the angular acceleration is zero $\alpha = 0$, and the tangential acceleration is zero as well. Since $v = \omega r$, we find for the centriptal acceleration towards the center, $a_c = v_t^2/r = (r\omega)^2/r$, or

$$a_c = r\omega^2. \tag{70}$$

For rolling motion of a wheel without slipping, the linear speed of the center must be the same as tangential speed: $v_{\mathrm{CM}} = r\omega$.

8.2 Mechanical Equilibrium



We consider a meter stick that is free to rotate about a fixed axis (at the end). Intuition tells us that the applied force $\vec{F}$ is more "effective" the greater the distance *L* from the axis, and the greater the angle $\theta < 90°$. For a quantitative definition, we draw a straight line through the force called the *line of action*. We then draw a line perpendicular to the line of action through the axis: the distance between the axis and the line of action is the lever arm,

$$l = L\sin\theta. \tag{71}$$

The magnitude of the torque is then

$$|\tau| = Fl = FL\sin\theta. \tag{72}$$

The torque is a vector quantity. For rotation about a fixed axis, the torque is either positive or negative for counter-clockwise- and clockwise rotation, respectively,

$$\tau \begin{cases} > 0 & \text{counterclockwise,} \\ < 0 & \text{clockwise.} \end{cases} . \tag{73}$$

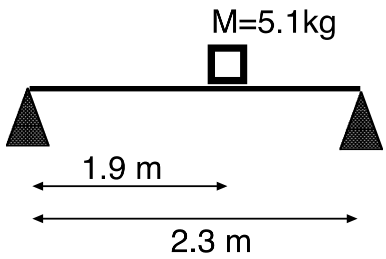
The unit of torque is force times distance, $[\tau] = [F] \cdot [l] = \mathrm{N} \cdot \mathrm{m}$. Note that $\mathrm{N} \cdot \mathrm{m} = \mathrm{J}$, but this is not used since torque and work are different physical quantities.

For a point mass, the condition $\sum \vec{F} = 0$ is sufficient to guarantee that an object initially at rest, stays in rest. For an extended object, the condition $\sum \vec{F} = 0$ only guarantees that there there is no translational motion, i.e., motion to the left and right, up and down, and into- and out-of the page. However, the object could still rotate. In general, the torque depends of the axis of rotation. In the case when the net force is zero, $\sum \vec{F} = 0$, the torque is independent of the choice of the axis. It follows that the condition $\sum \tau = 0$ is independent of the choice of the axis of rotation. We thus have the condition for mechanical equilibrium,

$$\sum \vec{F} = 0 \quad \text{and} \quad \sum \tau = 0 \qquad \text{(mechanical equilibrium)}. \tag{74}$$

At the introductory physics level, we solve restricted problems where the axis of rotation is out-of or into- the page, and all forces are in the page's plane. In this case, the condition for mechanical equilibrium consists of three equations so that we have at most three unknown quantities. Typical cases are: (1) three unknown forces in known directions [e.g., along a cable], and (2) one unknown force in a known direction and one unknown force in an unknown direction. Many practical problems are thus outside the scope of introductory physics; we cannot find the forces in each of the four legs of a table when an object sits on it in an arbitrary location.

In many cases, the equation for the net torque $\sum \tau = 0$ can be simplified by making a "smart" choice of the axis. For example, the lever arm for one or two forces is zero and the corresponding forces do not contribute to the net torque.

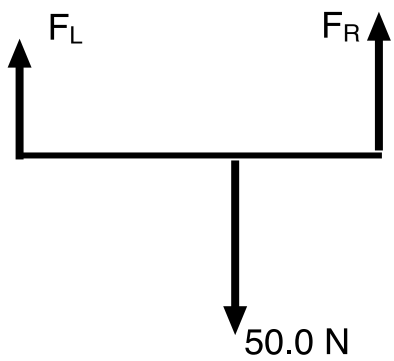


Example 3: A block with mass M = 5.1 kg sits on a massless plank of length L = 2.3 m. The plank is supported on its left and right ends by cones. The block sits 1.9-m away from the left cone.

a) Draw the appropriate free-body diagram for the problem. **b)** (i) Write down Newton's second law for the plank; and (ii) choose an axis of rotation and write down the torque about that axis. **c)** Find the forces that the cones exert on the plank. **d)** If the support forces by the left and right cones are in a ratio of two-to-one, i.e., $F_L/F_R = 2$, where is the block placed?

Solution: Note $W = Mg = 5.1\,\mathrm{kg} \cdot 9.8\,\mathrm{m/s^2} = 50.0\,\mathrm{N}$, we have

(i) We have Newton's second law,



$$\sum F = F_L + F_R - 50.0\,\mathrm{N} = 0;$$

(ii) we choose the left end of the plank as the axis of rotation, so that

$$\sum \tau = -50.0\,\mathrm{N} \cdot 1.9\,\mathrm{m} + F_R \cdot 2.3\,\mathrm{m} = 0.$$

We have

$$F_R = \frac{50.0\,\mathrm{N} \cdot 1.9\,\mathrm{m}}{2.3\,\mathrm{m}} = 41.3\,\mathrm{N},$$

so that

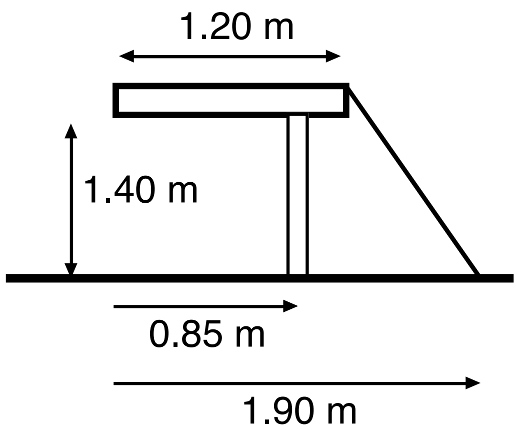
$$F_L = 50.0\,\mathrm{N} - 41.3\,\mathrm{N} = 8.7\,\mathrm{N}.$$

In the case $F_L = 2F_R$ so that

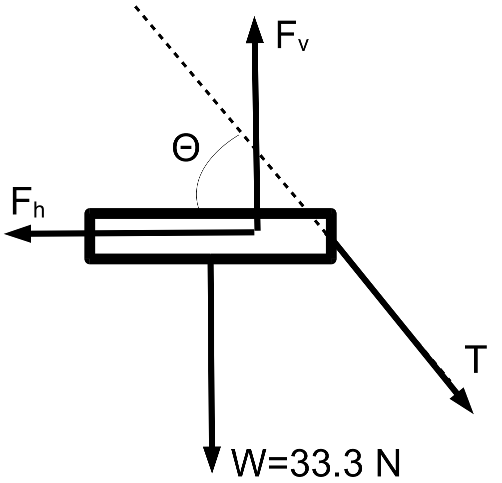
$$F_L + F_R = 3F_R = 50.0\,\mathrm{N} \quad \longrightarrow \quad F_R = \frac{50.0\,\mathrm{N}}{3} = 16.7\,\mathrm{N}.$$

We use x for the distance of the block from the left side of the plank. Then,

$$\tau = -50.0\,\mathrm{N} \cdot x + 16.7\,\mathrm{N} \cdot 2.3\,\mathrm{N} = 0 \quad \longrightarrow \quad x = \frac{16.7\,\mathrm{N} \cdot 2.3\,\mathrm{m}}{50.0\,\mathrm{N}} = 0.77\,\mathrm{m}.$$



Example 4: A wooden board with length $L = 1.20\,\mathrm{m}$ and mass $M = 3.4\,\mathrm{kg}$ is supported by one leg at the distance $d = 0.85\,\mathrm{m}$ from the left end of the board, and a cable anchored to the floor at 1.9 m from the left end of the board. The length of the leg is 1.4 m. **a)** Draw the free-body diagram for the problem. **b)** Write down Newton's second law for the board. **c)** Choose an axis of rotation and write down the torque about that axis. **d)** Find the tension in the cable. **e)** What is the magnitude of the force exerted by the leg on the board?



Solution: The angle of the cable with respect to the horizontal is $\tan\theta = (1.4\,\mathrm{m})/(0.7\,\mathrm{m}) = 2$, so that $\theta = 63.4°$. We have the the weight of the table $W = Mg = 33.3\,\mathrm{N}$; the forces from the leg F_h and F_v and the tension on the cable T are determined by

$$\sum F_x = F_h - T\cos 63.4° = 0,$$

$$\sum F_y = -T\sin 63.4° + F_v - 33.3\,\mathrm{N} = 0.$$

We choose the point of contact of the leg with the board as the axis of rotation. Then,

$$\sum \tau = 33.3\,\mathrm{N} \cdot 0.25\,\mathrm{m} - T \cdot 0.35\,\mathrm{m}\ \sin 63.4^\circ = 0.$$

Since $0.35\,\mathrm{m} \cdot \sin 63.4^\circ = 0.31\,\mathrm{m}$, we have the equation for the torque,

$$T = \frac{8.3\,\mathrm{N} \cdot \mathrm{m}}{0.31\,\mathrm{m}} = 26.8\,\mathrm{N}.$$

We see

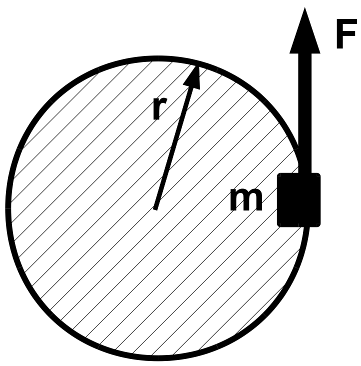
$$F_v = 33.3\,\mathrm{N} + 26.8\,\mathrm{N} \cdot \sin 63.4^\circ = 57.3\,\mathrm{N}, \qquad F_h = 26.8\,\mathrm{N} \cdot \cos 63.4^\circ = 12.0\,\mathrm{N}.$$

We thus get for the magnitude of the force exerted by the leg,

$$F_{\mathrm{leg}} = \sqrt{F_h^2 + F_v^2} = \sqrt{(12.0\,\mathrm{N})^2 + (57.3\,\mathrm{N})^2} = 58.5\,\mathrm{N}.$$

8.3 Rotational Dynamics

We consider the simple case when an object with mass m sits on the perimeter of a massless disk with radius r. The general case for an extended object can be obtained by adding point masses [superposition principle].



The disk can rotate freely about the center. A force F acts on the mass m in a tangential direction. The torque extered by the force about the axis, $\tau = Fr = (ma)r$, where we used $F = ma$. Since a is the tangential acceleration, we have $a = \alpha r$, so that $\tau = m(\alpha r)r = (mr^2)\,\alpha$. We define the motion of inertia

$$I = \sum_i m_i r_i^2. \tag{75}$$

The moment of inertia of an extended object about a fixed axis is usually given. The equation of motion is then

$$\tau = I\alpha. \tag{76}$$

This is Newton's second law for rotation: the force corresponds to the torque $(F \leftrightarrow \tau)$, the mass corresponds to the moment of inertia $(m \leftrightarrow I)$, and the linear acceleration corresponds to the angular acceleration $(a \leftrightarrow \alpha)$.

When the disk rotates through the angle θ, the arc length is $s = r\theta$, so that the force is doing work on the mass $W = Fs = Fr\theta$. Since $\tau = Fr$ is the torque, we have for the work,

$$W = \tau\theta. \tag{77}$$

If the object starts from rest, $v_0 = 0$, the speed of the object follows from the work-kinetic energy theorem, $W = \mathrm{KE} = mv^2/2$. The tangential speed is given by $v = \omega r$, and we get $W = m(\omega r)^2/2 = (mr^2)\omega^2/2$. We define the rotational kinetic energy,

$$\mathrm{KE}_{\mathrm{rot}} = \frac{1}{2}I\omega^2. \tag{78}$$

This yields the work-kinetic energy theorem for rotation,

$$W = \Delta\mathrm{KE}_{\mathrm{rot}}. \tag{79}$$

When there is no net torque $W = 0$, the rotational kinetic energy is constant.

If the force acts during the time interval Δt, the impulse $J = F\Delta t$ changes the momentum of the object $J = F\Delta t = mv$ for $v_0 = 0$ at time $t = 0$. We write $(Ft) \cdot r = (Fr) \cdot t = (mv) \cdot r = (mr^2)\omega t$, and define the angular momentum,

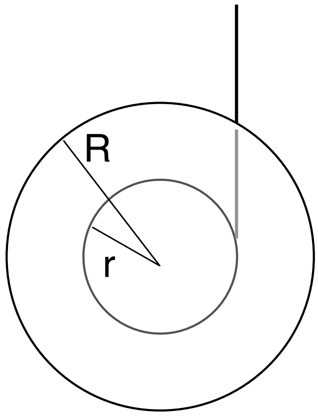
$$L = \sum_i p_i r_i = I\omega. \tag{80}$$

Thus, Netwon's second law for rotation relates to the change in the angular momentum to the rotational impulse,

$$J_{\mathrm{rot}} = \tau \cdot \Delta t = \Delta L. \tag{81}$$

We compare translational and rotational dynamics in this table:

Translation	Rotation
Mass m	Moment of inertia $I = \sum m_i r_i^2$
Force F	Torque $\tau = \sum F_i r_i$
Acceleration a	Angular acceleration α
Newton's 2nd law $F = ma$	"Newton's 2nd law" $\tau = I\alpha$
Momentum $p = mv$	Angular momentum $L = pr = mvr$
$\sum F = 0$: $p_{\text{total}} = \text{const}$	$\sum \tau = 0$: $L_{\text{rot}} = \text{const}$
Kinetic energy $\text{KE} = mv^2/2$	Rotational kinetic energy $\text{KE}_{\text{rot}} = I\omega^2/2$
Work $W = Fs$	Work $W = \tau\Delta\theta$
Work-KE $W = \Delta\text{KE}$	Work-KE $W = \Delta\text{KE}_{\text{rot}}$



Example 5: A Yo-Yo is a uniform disk with mass $m = 0.24\,\mathrm{kg}$, outer radius $R = 0.12\,\mathrm{m}$, and inner radius $r = 0.08\,\mathrm{m}$. A string is wrapped around the inner ring. The string hangs from the ceiling and the Yo-Yo falls under the influence of it own weight. **a)** The Yo-Yo starts to fall from rest. After the Yo-Yo falls from the height $h = 0.20\,\mathrm{m}$, it travels with the speed $v = 0.6\,\mathrm{m/s}$. Find the angular velocity and acceleration at that instant. **b)** Find the moment of inertia of the Yo-Yo. **c)** Find the tension in the string.

Solution: We have for the angular speed,

$$\omega = \frac{v}{r} = \frac{0.6\,\mathrm{m/s}}{0.08\,\mathrm{m}} = 7.5\,\frac{\mathrm{rad}}{\mathrm{s}}.$$

We have the angular displacement $\Delta\theta = h/r = 0.20\,\mathrm{m}/0.08\,\mathrm{m} = 2.5\,\mathrm{rad}$. Since $\omega_0 = 0$, the angular acceleration is

$$\omega^2 = 2\alpha\Delta\theta \qquad \longrightarrow \qquad \alpha = \frac{\omega^2}{2\Delta\theta} = \frac{(7.5\,\mathrm{rad/s})^2}{2\cdot 2.5\,\mathrm{rad}} = 11.25\,\frac{\mathrm{rad}}{\mathrm{s}^2}.$$

At the start, the Yo-Yo has gravitational potential energy,

$$\mathrm{PE} = mgh = 0.24\,\mathrm{kg}\cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2}\cdot 0.20\,\mathrm{m} = 0.47\,\mathrm{J}.$$

The gravitational energy is transformed into kinetic energy for translation and rotation. The translational kinetic energy is

$$\mathrm{KE_{trans}} = \frac{1}{2}0.24\,\mathrm{kg}\cdot\left(0.6\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 0.04\,\mathrm{J}.$$

Thus, the rotational kinetic energy is

$$\mathrm{KE_{rot}} = \mathrm{PE} - \mathrm{KE_{trans}} = 0.47\,\mathrm{J} - 0.04\,\mathrm{J} = 0.43\,\mathrm{J} = \frac{1}{2}I\cdot\omega^2,$$

and

$$I = \frac{2 \cdot \mathrm{KE_{rot}}}{\omega^2} = \frac{2 \cdot 0.43\,\mathrm{J}}{(7.5\,\mathrm{rad/s})^2} = 1.53 \times 10^{-2}\,\mathrm{kg} \cdot \mathrm{m}^2.$$

We have the torque exerted on the Yo-Yo,

$$\tau = I\alpha = 1.53 \times 10^{-2}\,\mathrm{kg\,m^2} \cdot 11.25\,\frac{\mathrm{rad}}{\mathrm{s}^2} = 0.17\,\mathrm{N\,m}.$$

For the tension T,

$$\tau = Tr \quad \longrightarrow \quad T = \frac{\tau}{r} = \frac{0.17\,\mathrm{N\,m}}{0.08\,\mathrm{m}} = 2.2\,\mathrm{N}.$$

Example 6: The moment of inertia of the carousel with radius $R = 4.5\mathrm{m}$ is $I_{\mathrm{carousel}} = 1874\,\mathrm{kg\ m}^2$. Emmy has mass $m = 56\,\mathrm{kg}$, and stands on the carousel at the radius $r_0 = 4.0\,\mathrm{m}$. **a)** The carousel completes a full revolution every 25 seconds. Find the *total* angular momentum and the *total* rotational kinetic energy of the entire system [carousel *plus* Emmy]. **b)** She now walks towards the center and stops at the radius $r = 3.2\,\mathrm{m}$. How long does it take to complete a full revolution when she is at $r = 3.2\,\mathrm{m}$? **c)** The carousel completes one quarter of a revolution, while she walks from $r_0 = 4.0\,\mathrm{m}$ to $r = 3.2\,\mathrm{m}$. Find the (average) torque exerted on the carousel by Emmy.

Solution: We have to add Emmy's moment of inertia to the carousel's moment of inertia. The total moment of inertia is thus

$$I_{\mathrm{total}} = I_{\mathrm{carousel}} + mr_0^2 = 1874\,\mathrm{kg\,m^2} + 56\,\mathrm{kg} \cdot (4.0\,\mathrm{m})^2 = 2770\,\mathrm{kg\,m^2}.$$

We have $\omega = (2\pi\,\mathrm{rad})/25\,\mathrm{s} = 0.25\,\mathrm{rad/s}$, and for the angular momentum,

$$L_{\mathrm{total}} = I_{\mathrm{total}}\,\omega = 2770\,\mathrm{kg\,m^2} \cdot 0.25\,\frac{\mathrm{rad}}{\mathrm{s}} = 692.5\,\frac{\mathrm{kgm^2}}{\mathrm{s}};$$

for the rotational kinetic energy,

$$\mathrm{KE_{rot}} = \frac{1}{2} I_{\mathrm{total}}\,\omega^2 = \frac{1}{2} 2770\,\mathrm{kg\,m^2} \cdot \left(0.25\,\frac{\mathrm{rad}}{\mathrm{s}}\right)^2 = 86.6\,\mathrm{J}.$$

Emmy's moment of inertia is changing as she walks towards the center. We now have

$$I'_{\mathrm{total}} = I_{\mathrm{carousel}} + mr^2 = 1874\,\mathrm{kg\,m^2} + 56\,\mathrm{kg} \cdot (3.2\,\mathrm{m})^2 = 2447\,\mathrm{kg\,m^2}.$$

The total angular moment is conserved while she walks. We get the new angular velocity ω',

$$L_{\mathrm{total}} = I_{\mathrm{total}}\,\omega = I'_{\mathrm{total}}\,\omega' \quad \longrightarrow \quad \omega' = \frac{I_{\mathrm{total}}}{I'_{\mathrm{total}}}\,\omega = \frac{2770\,\mathrm{kg\,m^2}}{2448\,\mathrm{kg\,m^2}} \cdot 0.25\,\frac{\mathrm{rad}}{\mathrm{s}} = 0.28\,\frac{\mathrm{rad}}{\mathrm{s}}.$$

We also get for the period, $T' = 2\pi/\omega' = 2\pi/(0.28\,\mathrm{rad/s}) = 22.4\,\mathrm{s}$. We use the work-kinetic energy theorem $W = \Delta\mathrm{KE}_{\mathrm{rot}}$ to calculate the work done by Emmy on the carousel. Since $\mathrm{KE}_{\mathrm{rot}} = \frac{1}{2}I_{\mathrm{carousel}}\omega^2$, we have

$$W = \frac{1}{2}I_{\mathrm{carousel}}\left[(\omega')^2 - \omega^2\right] = \frac{1}{2}1874\,\mathrm{kg\,m}^2 \cdot \left[\left(0.28\,\frac{\mathrm{rad}}{\mathrm{s}}\right)^2 - \left(0.25\,\frac{\mathrm{rad}}{\mathrm{s}}\right)^2\right] = 14.9\,\mathrm{J}.$$

For the angular displacement we have $\Delta\theta = 2\pi\,\mathrm{rad}/4 = 1.57\,\mathrm{rad}$. Since $W = \tau_{\mathrm{ave}}\Delta\theta$,

$$\tau_{\mathrm{ave}} = \frac{W}{\Delta\theta} = \frac{14.9\,\mathrm{J}}{1.57\,\mathrm{rad}} = 9.5\ \mathrm{N}\cdot\mathrm{m},$$

or about $7\,\mathrm{lb}\cdot\mathrm{ft}$ [“foot pound”].

Example 7: A solid cylinder with mass $m = 3.2\,\mathrm{kg}$ and radius $r = 3.6\,\mathrm{cm}$ rolls down an incline at an angle $\theta = 23°$ with length $s = 1.9\,\mathrm{m}$. Assume that the cylinder rolls without slipping. **a)** Find the speed of the cylinder at the bottom of the ramp. **b)** Find the torque acting on the cylinder.

Solution: Because there is only static and no kinetic friction, mechanical energy is conserved: gravitational potential energy is converted into translational and rotational kinetic energy,

$$mgh = \frac{m}{2}v^2 + \frac{I}{2}\omega^2,$$

with $h = 1.9\,\mathrm{m}\sin 23° = 0.74\,\mathrm{m}$. Because the cylinder rolls without slipping, $\omega = v/r$, so that

$$mgh = \frac{1}{2}\left[m + \frac{I}{r^2}\right]v^2.$$

We find for the speed,

$$v = \sqrt{\frac{2mgh}{m + I/r^2}} = \sqrt{\frac{2gh}{1 + I/mr^2}}.$$

Since $v = \sqrt{2gh}$, the speed of the cylinder at the end of the ramp is slower than the speed of the mass after sliding down the incline. For a cylinder $I = mr^2/2$ so that $I/mr^2 = 1/2$. We then have

$$v = \sqrt{\frac{2gh}{1 + 1/2}} = \sqrt{\frac{4}{3}\cdot 9.8\,\mathrm{m/s}\cdot 0.74\,\mathrm{m}} = 3.1\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We note that the speed only depends on the radius of the cylinder, not its mass. For the acceration of the cylinder along the incline, we get $a = v^2/2s = (3.1\,\mathrm{m/s})^2/(2\cdot 1.9\,\mathrm{m}) = 2.5\,\mathrm{m/s}^2$. The angular acceleration follows

$$\alpha = \frac{a}{r} = \frac{2.5\,\mathrm{m/s}^2}{0.036\,\mathrm{m}} = 69.4\,\frac{\mathrm{rad}}{\mathrm{s}^2}.$$

The moment of inertia is $I = 3.2\,\mathrm{kg}\cdot(0.036\,\mathrm{m})^2/2 = 2.1\times 10^{-3}\,\mathrm{kg\,m}^2$, so the torque is,

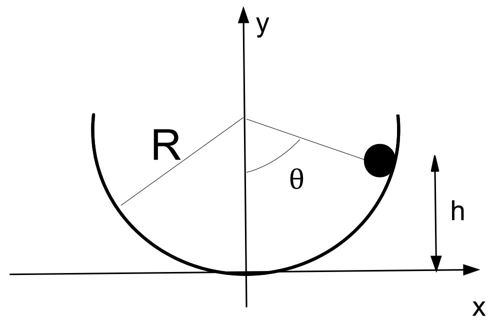
$$\tau = I\alpha = 2.1\times 10^{-3}\,\mathrm{kg\,m}^2\cdot 70.2\,\frac{\mathrm{rad}}{\mathrm{s}} = 0.146\,\mathrm{N\,m}.$$

The torque is produced by the static friction force $\tau = f_s r$, such that the friction force follows,

$$f_s = \frac{\tau}{r} = \frac{0.145\,\mathrm{N\,m}}{0.036\,\mathrm{m}} = 4.0\,\mathrm{N}.$$

The maximum static friction force is given by $f_{s,\mathrm{max}} = \mu_s F_N$, where
$F_N = 3.2\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s}^2\cdot\cos 23° = 28.9\,\mathrm{N}$. We find the coefficient of static friction $\mu_s = f_s/F_N = 0.14$.

9 Oscillations



Oscillations describe motions that recur after a fixed time, or a period T so that for any integer n,

$$x(t+T) = x(t), \quad v(t+T) = v(t), \quad a(t+T) = a(t).$$

That is, the object returns to its starting point after a period. A pearl moving in a bowl is a simple example of harmonic motion. The pearl is initially at rest and is placed at the height h, so that it is displaced at the angle $\cos\theta_0 = (1 - h/R)$ with an initial speed $\omega_0 = 0$. For $t > 0$, the angle decreases, so that the angular velocity is negative $\omega = \Delta\theta/\Delta t < 0$. When the pearl is at the bottom of the bowl $\theta = 0$, the speed of the pearl is a maximum] $\omega = -|\omega|_{\max}$. The pearl then continues to move and rolls upwards on the other side of the bowl $\theta < 0$, until it reaches a minimum $\theta_{\min} = -\theta_0$. At the turning point, the velocity is zero $w = 0$. The pearl then travels back towards the bottom of the bowl, so that $w > 0$. The pearl rolls upwards towards the initial angle ϕ_0, and the motion repeats itself.

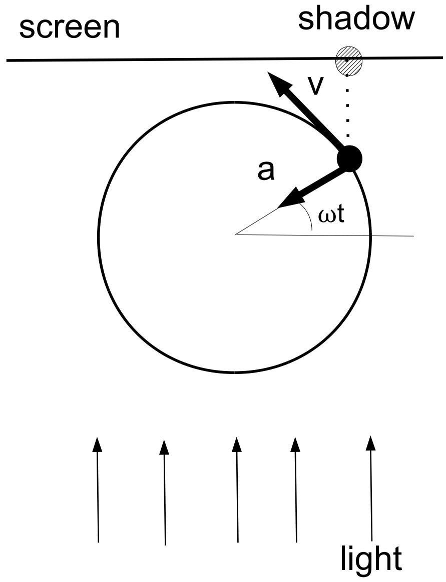
9.1 Energy Consideration

From the description of the pearl, we have have for the energy

Time	KE	PE
0	mgh	0
$T/4$	0	$mv_0^2/2$
$T/2$	mgh	0
$3T/4$	0	$mv_0^2/2$
T	mgh	0

That is, potential energy is transformed into kinetic energy, which is then transformed back into potential energy, $\mathrm{PE} \longrightarrow \mathrm{KE} \longrightarrow \mathrm{PE} \longrightarrow \ldots$. The period of energy transformation is twice that of the coordinate $T_E = 2T$. The transformation of two forms of energy into each other is common for all harmonic motion. The maximum angle ϕ_0 is the amplitude of harmonic motion. Since $v_0 = \sqrt{2g(1 - \cos\theta_0)}$, the amplitude also determines the maximum speed of the oscillator [e.g., the pearl].

9.2 Motion on the Reference Circle: Simple Harmonic Motion



We consider the uniform motion of a point along a circle of radius *A*. Light produces a shadow on a screen. The motion of the shadow mimics harmonic motion, and angular velocity is related to the period, $\omega = 2\pi/T$, cf. Eq. (62). The coordinate of the shadow is $x = A\cos(\omega t)$. The magnitude of the object's velocity on the reference circle is constant $v = A\omega$. The component of the velocity vector along the screen is $-A\omega\,\sin(\omega t)$. The magnitude of the (centriptal) acceleration of the object is $a_c = (A\omega)^2/A = A\omega^2$ so that the acceleration of the shadow is $a = -A\omega^2\cos(\omega t)$. Since $a = -\omega^2 \cdot A\cos(\omega t)$, we find,

$$a = -\omega^2 x. \tag{82}$$

Since $F = ma$, the force follows,

$$F = -m\omega^2 x. \tag{83}$$

That is, the force is linear in the coordinate x. The force is negative, $F < 0$ [$F < 0$] when $x > 0$ [$x < 0$]; that is, the force drives the shadow back to the origin $x = 0$. We say that *F* is a *linear restoring force.* The period of oscillatory motion is independent of the amplitude of the motion.

Example 1: A pearl is moving in a bowl. **a)** Show that the force on the pearl is a linear for small angular displacements. **b)** Determine the period of harmonic motion of the pearl.

Solution: The weight of the pearl in tangential direction is given by $F_{\text{tangential}} = -mg\sin\phi$. For small angles [in radians], $\sin\phi \simeq \phi$, so that the force is given by

$$F_{\text{tangential}} \simeq -mg\phi.$$

That is, the force is proportional to the angular displacement. We write the tangential acceleration in terms of angular acceleration, $a = R\alpha$, where R is the radius of the bowl. Newton's second law yields $mR\alpha = -mg\phi$, or

$$\alpha = -\frac{g}{R}\phi.$$

Comparison with Eq. (83) then yields $\omega = \sqrt{g/R}$, or for the period

$$T = 2\pi\sqrt{\frac{R}{g}}.$$

Discussion: The condition of small angular displacement is not as limited as it may appear. For $\phi = \pi/6$ [or $\phi = 30°$], we have $\sin\phi = 0.5$ or $\sin\phi \simeq \phi$ within 5%. For a bowl of radius $R = 20$ cm, the period follows $T = 2\pi\sqrt{(0.2\,\mathrm{m})/(9.8\,\mathrm{m/s^2})} = 0.9\,\mathrm{s}$, which is easily verified at home.

We assume above that $x = A$ at time $t = 0$. This is not the case in general, and instead we have for the 'phase angle' $\phi = \omega t \longrightarrow \phi_0 + \omega t$. We then have for the coordinate, velocity, and acceleration:

$$x(t) = A\cos(\omega t + \phi_0), \tag{84}$$

$$v(t) = -A\omega\sin(\omega t + \phi_0), \tag{85}$$

$$a(t) = -A\omega^2\cos(\omega t + \phi_0). \tag{86}$$

These equations can also be derived with methods from calculus [i.e., taking the derivates with respect to time t]. The oscillatory motion is specified by the amplitude A and the initial phase ϕ_0.

Example 2: At time $t = 0$, the displacement is $x = 2.72\,\mathrm{m}$, the velocity is $v = -2.54\,\mathrm{m/s}$, and the acceleration is $a = -10.87\,\mathrm{m/s^2}$. **a)** Find the amplitude, phase, and angular frequency for the harmonic motion. **b)** Find expressions for $x(t)$, $v(t)$, and $a(t)$.

Solution: We find for the ratio of the acceleration and coordinate,

$$\frac{a}{x} = \frac{-10.87\,\mathrm{m/s^2}}{2.72\,\mathrm{m}} = -4.0\,\mathrm{s^{-2}} = -\omega^2 \quad \longrightarrow \quad \omega = 2.0\,\mathrm{s^{-1}},$$

and the period is $T = 2\pi/\omega = 2\pi/(2.0\,\mathrm{s^{-1}}) = 3.1\,\mathrm{s}$.

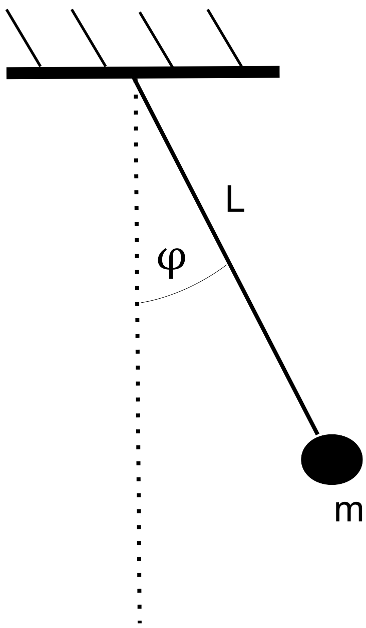
Since $v^2 = (A\omega)^2 \sin^2(\omega t + \phi_0) = (A\omega)^2[1 - \cos^2(\omega t + \phi_0)]$, we get $v^2 + (\omega x)^2 = (A\omega)^2$. We have

$$v^2 + (\omega x)^2 = \left(-2.54\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + \left(2.0\,\frac{1}{\mathrm{s}} \cdot 2.72\,\mathrm{m}\right)^2 = 36\,\left(\frac{\mathrm{m}}{\mathrm{s}}\right)^2,$$

so that for the amplitude $A = (6.0\,\mathrm{m/s})/(2.0\,\mathrm{s^{-1}}) = 3.0\,\mathrm{m}$. Since $x = A\cos\phi_0$ at time $t = 0$,

$$\cos\phi_0 = \frac{x}{A} = \frac{2.72\,\mathrm{m}}{3.0\,\mathrm{m}} = 0.9 \quad \longrightarrow \quad \phi_0 = 0.43\,\mathrm{rad},$$

or $\phi_0 = 25°$.



Example 3: Find the period of a mathematical pendulum that is an object with mass m attached to a string of length L.

Solution: The force on the object is the weight $F = mg$ (directed downward). For the torque about the fix point on the 'ceiling,' we get

$$\tau = -mg \cdot L \sin \phi.$$

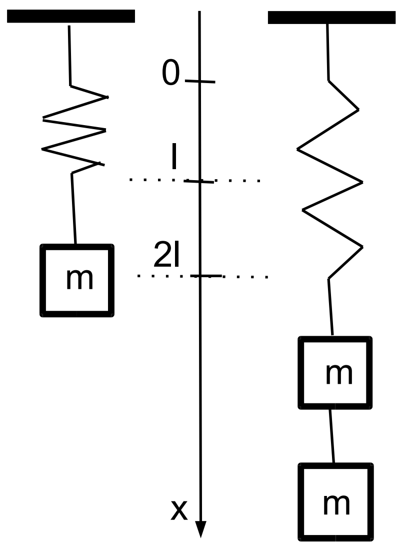
Note that the torque is negative (clockwise) [positive (counter-clockwise)] when the angular displacement is positive, $\phi > 0$ [negative $\phi < 0$]. Since the moment of inertia is $I = mL^2$, we have $\tau = I\alpha$ so that $-mgL \sin \alpha = mL^2 \sin \phi$, or $\alpha = -\frac{g}{L}\phi$.

Comparison with Eq. (82) yields

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

Discussion: We find the length of a mathematical pendulum with period $T = 2.0\,\mathrm{s}$. We get $L = (9.8\,\mathrm{m/s^2})/(\pi\,\mathrm{s})^2 = 1.0\,\mathrm{m}$.

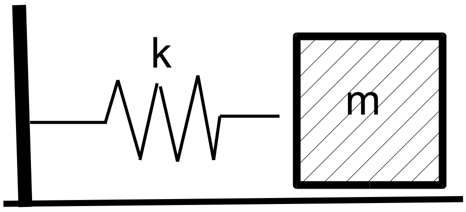
9.3 Elastic Forces: Spring



An elastic spring is attached to the ceiling. A block with mass m is attched to the spring, and the spring is stretched by the distance l. If two blocks with total mass $2m$ are attached, the spring is displaced by the distance $2l$. The forces on the block are the weight mg [downward] and the elastic force [upwards]. Because the block is in mechanical equilibrium, the net force on the block is zero. We conclude that the elastic force is a linear restoring force,

$$F_{\text{elast}} = -kx, \tag{87}$$

where k is the spring constant with unit $[k] = [F]/[x] = \mathrm{N/m}$. We thus have $F > 0$ for $x < 0$ and $F < 0$ for $x > 0$.



We consider a block with mass m sliding on a frictionless horizontal surface. The force along the horizontal direction is given by $F = F_{\text{elast}} = -kx$ so that the equation of motion of the block follows $F = ma = -kx$, or

$$a = -\frac{k}{m}x. \tag{88}$$

Comparison with Eq. (82) yields $\omega^2 = k/m$ so that

$$T = 2\pi\sqrt{\frac{m}{k}}. \tag{89}$$

Springs are stiffer for greater values of the spring constant. Eq. (89) then shows that stiffer springs oscillate with shorter periods.

Example 4: A block with mass $m = 0.76\,\text{kg}$ is attached to a spring hanging from the ceiling so that the spring stretched by 12 cm. Find the period of oscillations when the block is streteched 3 cm more.

Solution: We find the spring constant from $F = mg$ and $x_0 = 0.12\,\text{m}$:

$$k = \frac{mg}{x_0} = \frac{0.76\,\text{kg}\cdot 9.8\,\text{m/s}^2}{0.12\,\text{m}} = 62.1\,\frac{\text{N}}{\text{m}}.$$

For $x = 0.15\,\text{cm} = x_0 + \Delta x$ with $\Delta x = 0.03\,\text{m}$, we have from Newton's second law:

$$ma = -kx + mg = -k\Delta x - kx_0 + mg = -k\Delta x.$$

We thus find the period:

$$T = 2\pi\sqrt{\frac{0.76\,\text{kg}}{62.1\,\text{N/m}}} = 0.7\,\text{s}.$$

Elastic Potential Energy: In Eq. (37), we defined change in potential energy as the negative of the work done on a system. We stretch a spring from x to $x + \Delta x$ by applying the external force $F_{\text{ext}} = -F_{\text{elastic}} = kx$. Thus the work done by the elastic work is $W_{\text{elastic}} = F_{\text{elastic}} \cdot \Delta x = -kx\Delta x$. If we stretch the spring from $x = 0$ to a finite value x, the average elastic force is $F_{\text{elastic,ave}} = (0 - kx)/2 = -kx/2$ so that the work done by the elastic force is $W_{\text{elastic}} = F_{\text{elastic,ave}} \cdot x = (-kx/2)\cdot x = -kx^2/2$. We thus find the elastic potnetial energy:

$$\text{EPE} = \frac{kx^2}{2}. \tag{90}$$

Example 5: A 2.5 kg-block is resting on a flat horizontal table. A horizontal spring with spring constant $k = 62\,\text{N/m}$ is attached to the block, as shown above. The block is displaced by 8.5 cm to the right and then released from rest. The block then begins to move to the *left*. **a)** Find the time that elapses until the block begins to move to the *right*. **b)** What is the it average speed of the block from the moment it is released until it begins to move to the right? **c)** What is the total energy of the system [i.e., the spring *plus* the mass]? **d)** Assume that the block is released from rest at time $t = 0$. Find the time when the elastic energy of the spring is three times the kinetic energy of the block .

Solution: The period of the oscillation is

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{2.5\,\text{kg}}{62\,\text{N/m}}} = 1.26\,\text{s}.$$

The time when the block is moving to the left is half the period: $t^* = T/2 = 0.63\,\mathrm{s}$. The total displacement is $\Delta x = 2A = 2 \cdot 0.085\,\mathrm{m} = 0.17\,\mathrm{m}$. Then

$$v_{\mathrm{ave}} = \frac{\Delta x}{t^*} = \frac{0.17\,\mathrm{m}}{0.63\,\mathrm{s}} = 0.27\,\frac{\mathrm{m}}{\mathrm{s}}.$$

The kinetic energy is zero when the block is released. The total mechanical energy follows

$$E_{\mathrm{tot}} = \mathrm{EPE} = \frac{1}{2}kA^2 = \frac{1}{2}62\,\frac{\mathrm{N}}{\mathrm{m}} \cdot (0.085\,\mathrm{m})^2 = 0.22\,\mathrm{J}.$$

When $\mathrm{EPE} = 3\mathrm{KE}$, the total mechanical energy follows $E_{\mathrm{mech}} = \mathrm{KE} + \mathrm{EPE} = 4\mathrm{KE}$, so that

$$\mathrm{KE} = \frac{1}{2}mv^2 = \frac{E_{\mathrm{tot}}}{4} = \frac{1}{8}kA^2 = \frac{1}{8}m\omega^2 A^2,$$

where we inserted $k = m\omega^2$. We find

$$v = \pm\frac{1}{2}\omega A.$$

Since $v(t) = -A\omega\sin(\omega t)$, we get $-A\omega\sin(\omega t) = -A\omega/2$ so that $\sin(\omega t) = 1/2$. We find

$$\omega t = 0.52\,\mathrm{rad}, \quad \longrightarrow \quad t = \frac{0.52\,\mathrm{rad}}{2\pi\,\mathrm{rad}/(1.26\,\mathrm{s})} = 0.104\,\mathrm{s}.$$

9.4 Resonance

When the block [mass m] sitting on the frictionless and attached to a spring [constant k] is released from $x_0 = A$ it undergoes oscillatory motion with angular frequency $\omega_0 = \sqrt{k/m}$; we refer to it as the "natural frequency." The system is driven when the block is moved by an external motor with angular frequency ω and amplitude a. If the motor is turned on at time $t = 0$, the block undergoes irregular motion for a certain time interval ["transient" interval] and then settles for a steady motion with angular frequency ω. The amplitude of the motion strongly depends on the frequency, $A = A(\omega)$: the largest amplitude occurs when the driving frequency coincides with the natural frequency $A_{\max} = A(\omega_0)$. This phenomenon universal for all oscillatory systems and is called *resonance*.

10 Fluids

At a microscopic level, many-body systems consists of atoms and molecules. The same microscopic system may have different *macroscopic* properties, depending on the *phase* of the system; most systems have three phases: solid, liquid, and gas. Gases and liquids are combined and referred to as 'fluid.' For $\mathrm{H_2O}$, they are called ice, water, and vapor. In solids and liquids, the distance between neighboring atoms (molecules) is roughly equal to their 'size,' about $0.2\,\mathrm{nm} = 2 \times 10^{-10}\,\mathrm{m}$; in gases, the typical distance is about 10 times greater, or $2\,\mathrm{nm} = 2 \times 10^{-9}\,\mathrm{m}$. In solids, the atoms (molecules) are in fixed positions, while they move freely in liquids and gases. This explains why solids are stiff and fluids can easily be deformed.

10.1 Fluids at Rest

The *density* of an object is defined as the ratio of mass m divided by its volume,

$$\rho = \frac{m}{V}. \tag{1}$$

The unit of density is $[\rho] = [m]/[V] = \text{kg/m}^3$. The density of liquids and solids are of the order of thousands of kilograms per cubic meter [$\rho \sim 10^3\,\text{kg/m}^3$] and is about a thousand times smaller [$\rho \sim 1\,\text{kg/m}^3$] for gases, which is consistent with the times greater distance between atoms (molecules). For H_2O, the respective densities are:

Phase	Density ρ [kg/m^3]
Ice	917
Water (4°C)	1000
Vapor (100°C)	0.6

H_2O is has many unusual properties: among them is the greater density in the liquid phase compared to the solid phase. Note that the density of vapor strongly depends on temperature and pressure [see below].

Example 1: The approximate density of the human body is close to that of (liquid) water $\rho \simeq 1000\,\text{kg/m}^3$. Find the volume of a person.

Solution: For $m = 70\,\text{kg}$ [or 155 lbs], so for the volume,

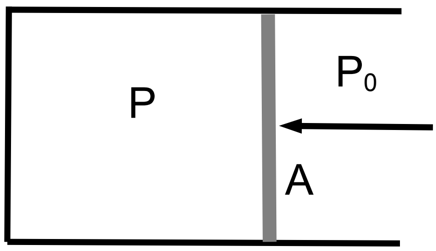
$$V = \frac{m}{\rho} = \frac{70\,\text{kg}}{1000\,\text{kg/m}^3} = 7.0 \times 10^{-2}\,\text{m}^3 = 7.0 \times 10^4\,\text{cm}^3.$$

This is equivalent to the volume of a cube with length $a \simeq 41\,\text{cm}$.

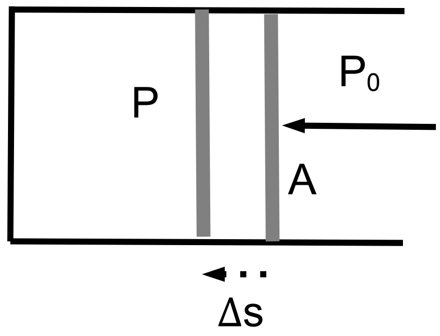
The response [e.g., the deformation] of a system to an external force F also depends on the size of the system. *Pressure* is defined as the ratio of force divided by the area,

$$P = \frac{F}{A}. \tag{2}$$

The SI unit of pressure is $[P] = [F]/[A] = \text{N/m}^2 = \text{Pa}$ ["Pascal"]. Normal atmospheric pressure is $P_{\text{atm}} = 1.015 \times 10^5\,\text{Pa}$. Pressure explains why we sleep in beds rather than on the floor: the forces exerted by the mattress and the floor on the person are equal [and equal to the weight of the person]. A mattress adjusts its shape to the person's body, thereby increasing the area and decreasing the pressure.



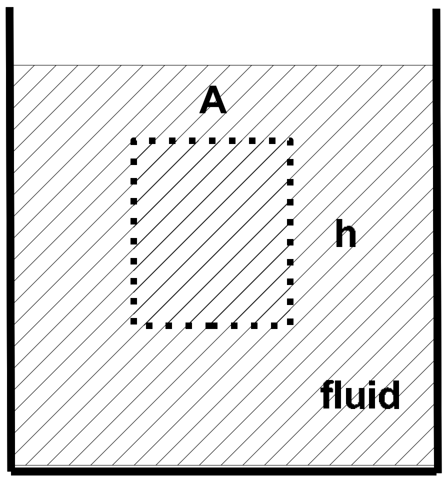
The pressure inside and outside a piston is P and $P_0 < P$, respectively. If the cross-sectional area of the piston is A, a force $F = (P - P_0) \cdot A$ must be applied to move the piston to the left.



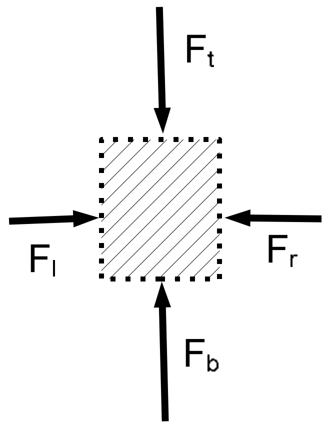
If the piston is displaced by Δs, work is done on the gas: $W = F \cdot \Delta s$. We find $W = (PA) \cdot \Delta s = P(A \cdot \Delta s) = -P\Delta V$, where we used $A\Delta s = -\Delta s$ [negative, because the volume of the gas *decreases*]. This yields an alternative definition of pressure

$$P = -\frac{W}{\Delta V}, \tag{3}$$

which is useful in many applications, especially when the change in the volume ΔV is small and the change in pressure can be ignored.



We examine the effect of gravity on a fluid. We consider a fluid element with a cross-sectional area A and a height h. The forces on the element are from the top F_{t}, the bottom F_{b}, the left F_l, and the right F_r.



There is no difference between left and right [since we cannot differentiate between the volume element and its mirror image] so that $F_l = F_r$. Since the volume element is in mechanical equilibrium, the difference between the force from the bottom and top is equal to the weight: $F_\mathrm{b} - F_\mathrm{t} = mg$. This difference is produced by the pressure difference $F_\mathrm{b} - F_\mathrm{t} = (P_\mathrm{b} - P_\mathrm{t})A$. The mass of the element is $m = \rho V = \rho A h$. We get $(P_\mathrm{b} - P_\mathrm{t})A = \rho A h g$, or $(P_\mathrm{b} - P_\mathrm{t}) = \rho g h$. If P_0 is the pressure on top of the surface, the pressure at the distance h below the surface is

$$P = P_0 + \rho g h. \tag{4}$$

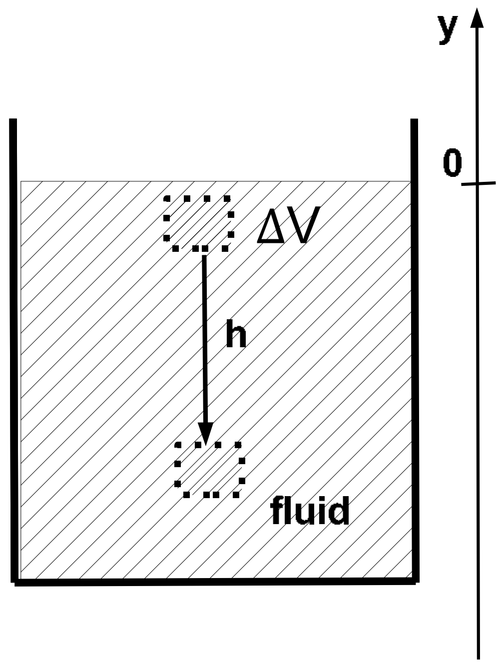
That is, the pressure increases with the depth below the surface of fluid at rest: *hydrostatic* pressure.

Example 2: Hydrostatic pressure is the basis of commonly used units of pressure. In medicine, blood pressure is reported as 120/80 [systolic/diastolic pressure]. The units are millimeter mercury. Compare atmospheric pressure to atmospheric pressure.

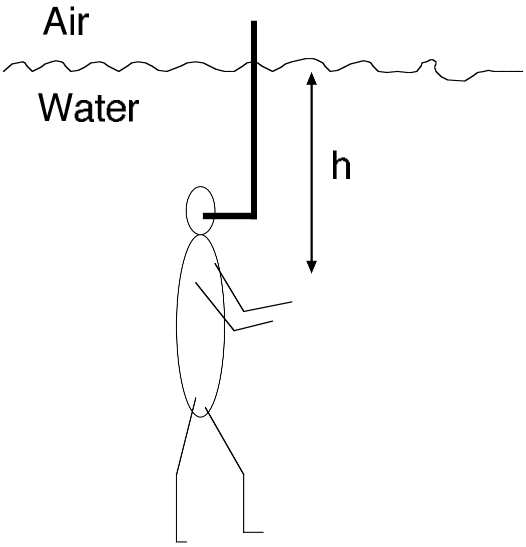
Solution: The density of mercury is $\rho_{\mathrm{Hg}} = 13,600\,\mathrm{kg/m}^3$. Thus

$$120\,\mathrm{mm\ Hg} = 0.12\,\mathrm{m} \cdot 13,600\,\frac{\mathrm{kg}}{\mathrm{m}^3} \cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} = 1.6 \times 10^4 \simeq \frac{1}{6}\,P_{\mathrm{atm}}.$$

The blood pressure produced by the heart is comparable to atmospheric pressure.



Alternatively, we consider a small fluid element with volume ΔV and mass $m = \rho\Delta V$. We imagine that we move the volume element slowly downward $\Delta y = -h$ so that the work done on the element is zero [using the work-kinetic energy theorem]. Because the work done by gravity is $W_g = mgh$, the external force does work $W = -W_g = -mgh$. We then get for the pressure associated with the vertical displacement $P = -(-mgh)/\Delta V = (m/\Delta V)gh = \rho gh$; that is, we recover the expression for the hydrostatic pressure. We choose a coordinate y along the vertical so that $y = 0$ at the surface. Then $P = P_0 + \rho h(-y)$, or $P_0 = P - \rho gy$: the sum of pressure and potential energy per volume is constant for a fluid element.



Example 3: A person is able to breath while diving with a snorkel. The flexible snorkel sticks out of the water. This is dangerous, however, because the lungs are compressed when a person is submerged. Estimate the maximum depth at which a person can safely snorkel.

Solution: The ribcage of a person can sustain the weight of a 100-lb person, so that $F \simeq 500\,\mathrm{N}$. The frontal area is $A = 0.3\,\mathrm{m} \times 0.3\,\mathrm{m} = 0.09\,\mathrm{m}^2$:

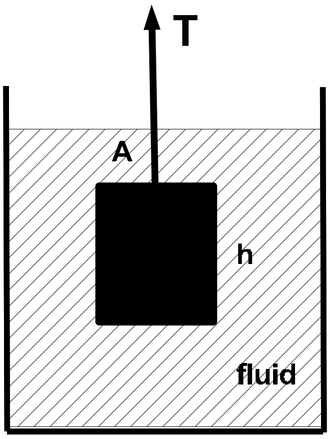
$$\Delta P = \frac{F}{A} = \frac{500\,\mathrm{N}}{0.09\,\mathrm{m}^2} = 5.5\,\mathrm{kPa}.$$

We set this pressure increase equal to the hydrostatic pressure:

$$\Delta P = \rho g h \quad \longrightarrow \quad h = \frac{\Delta P}{\rho g} = \frac{5.5\,\mathrm{kPa}}{1,000\,\mathrm{kg/m}^3 \cdot 9.8\,\mathrm{m/s}^2} = 0.56\,\mathrm{m};$$

Discussion: This estimate shows that snorkeling is limited to just below the surface.

Liquids do not have stiffness and any change in pressure applied to a completely enclosed fluid is transmitted undiminished to all parts of the fluid and the enclosing walls: *Pascal's principle*. If fluid container has two pistons with cross-sectional areas A_1 and $A_2 >> A_1$. A force F_1 is applied to piston 1. Since $P = F_1/A_1 = F_2/A_2$, the fluid "produces" the force $F_2 = (A_2/A_1)F_1$ to the piston 2. If $A_2 >> A_1$ then $F_2 >> F_1$. That is, a small force can produce a much larger one: this is the principle used in hydraulic lifts.



A block with mass m [cross-sectional area A and height h] is immersed in a fluid and suspended from a string with tension T. The tension is referred to as "apparent weight." For solid objects, the density of air is much smaller than the density of the object $\rho >> \rho_{\text{air}}$ and the apparent weight is very close to the weight mg. If the object is immersed in water [or a similar fluid], the apparent weight is less than the weight: $T < mg$. Because the net force on the block is zero, we have $T + \Delta P \cdot A = mg$, or $mg - T = \Delta P \cdot A > 0$. This is called the "buoyant force,"

$$F_B = \rho_{\text{fluid}} g V, \tag{5}$$

the buoyant force is equal to the weight of the displaced fluid: *Archimedes principle*.

Example 4: A person weighs 720 N in air and is lowered into a tank of water to about chin level. He now exhales as much air as possible and dunks his head underwater. His apparent weight while submerged is 34.3 N. **a)** Find his volume and his (average) density! **b)** Find the percentage of body fat (density of fat and lean body mass [soft tissue and bones]):

$$\rho_{\mathrm{f}} = 900\,\frac{\mathrm{kg}}{\mathrm{m}^3}, \quad \rho_{\mathrm{l}} = 1,100\,\frac{\mathrm{kg}}{\mathrm{m}^3}.$$

Solution: apparent weight W_a, buoyant force F_b and true weight W. Then $W_a + F_b - W = 0$, or $F_b = \rho_{\mathrm{water}} V_p g = W - W_a$ so that

$$V_{\mathrm{p}} = \frac{W - W_a}{\rho_{\mathrm{water}} g} = 7.0 \times 10^{-2}\,\mathrm{m}^3.$$

The density of the person follows

$$\rho_{\mathrm{p}} = \frac{W}{V_{\mathrm{p}} g} = 1050\,\frac{\mathrm{kg}}{\mathrm{m}^3}.$$

Assume that x_{f} is the fraction of fat so that $x_{\mathrm{l}} = (1 - x_{\mathrm{l}})$ is the fraction of the lean body mass. Then $x_{\mathrm{f}} M$ and $(1 - x_{\mathrm{f}})M$ are the mass of fat and lean body of the person. The volume of the person is

$$V = \frac{m_{\mathrm{f}}}{\rho_{\mathrm{f}}} + \frac{m_{\mathrm{l}}}{\rho_{\mathrm{l}}} = M\left[\frac{x_{\mathrm{f}}}{\rho_{\mathrm{f}}} + \frac{(1 - x_{\mathrm{f}})}{\rho_{\mathrm{l}}}\right].$$

The average density of the person follows $\rho_{\mathrm{p}} = \frac{M}{V} = [x_{\mathrm{f}}/\rho_{\mathrm{f}} + (1 - x_{\mathrm{f}})/\rho_{\mathrm{l}}]^{-1}$. We get

$$\left(\frac{1}{\rho_{\mathrm{p}}} - \frac{1}{\rho_{\mathrm{l}}}\right) = \left(\frac{1}{\rho_{\mathrm{f}}} - \frac{1}{\rho_{\mathrm{l}}}\right) x_{\mathrm{f}}.$$

Now solve for the fraction of fat:

$$x_{\mathrm{f}} = \frac{1}{\rho_{\mathrm{p}}}\left(\frac{\rho_{\mathrm{l}}\rho_{\mathrm{f}}}{\rho_{\mathrm{l}} - \rho_{\mathrm{f}}}\right) - \frac{\rho_{\mathrm{f}}}{\rho_{\mathrm{l}} - \rho_{\mathrm{f}}} = \frac{4950\,\mathrm{kg/m^3}}{\rho_{\mathrm{p}}} - 4.5.$$

The last equation is known as *Siri's formula*. For the person, we get the buoyant force: $F_b = 720\,\mathrm{N} - 34.3\,\mathrm{N} = 685.7\,\mathrm{N}$; then $F_b = \rho_w g V$ so that for the volume

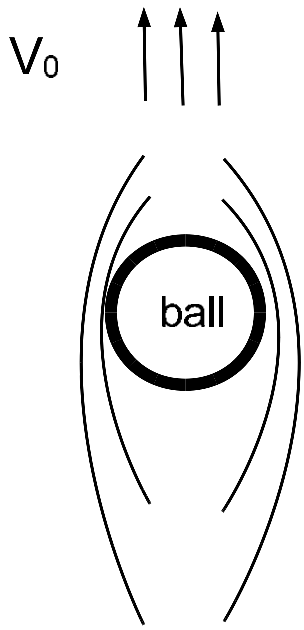
$$V = F_b/\rho_w g = 685.7\,\mathrm{N}/(1.0 \times 10^3\,\mathrm{kg/m^3} \cdot 9.8\,\mathrm{m/s^2}) = 7.0 \times 10^{-2}\,\mathrm{m}^3.$$

This give the (average) density of the person:

$$\rho_{\mathrm{p}} = m_{\mathrm{p}}/V = W_{\mathrm{air}}/(gV) = 720\,\mathrm{N}/(9.8\,\mathrm{m/s^2} \cdot 7.0 \times 10^{-2}\,\mathrm{m}^3) = 1,050\,\mathrm{kg/m^3}.$$

The fraction of fat follows $x_{\mathrm{f}} = (4950\,\mathrm{kg/m^3})/(1050\,\mathrm{kg/m^3}) - 4.5 = 4.714 - 4.5 = 0.214$, or $x_{\mathrm{f}} = 21.4\%$, which is reasonable for healthy adult male.

10.2 Fluids in Motion



A hair dryer can be used to hold up a light object [such as a *Ping-Pong* ball] in air. If $A = \pi R^2$ is the cross-sectional area of the ball, the pressure difference between the top and bottom is related to the weight of the ball: $(P_\mathrm{b} - P_\mathrm{t})A = mg$. We conclude that flow of air changes the pressure. Air flows from under the ball and comes to a (near) stop at the bottom. The air wraps around the edges and then recombines near the top, so that the flow speed is nonzero. This shows that the pressure is high (low) in regions with small (large) flow speeds.

We have seen for static fluids, cf. Eq. (6), that the sum of pressure and potential energy per volume is constant. For fluids in motion, we replace the potential energy by the total mechanical energy, that is, the sum of potential energy plus kinetic energy. We thus get

$$P + \rho g y + \frac{\rho}{2} v^2 = \mathrm{const.} \tag{6}$$

This is the *Bernoulli* equation. It shows that the pressure is reduced in regions where the fluid flow is fast, just as in the case of the *Ping-Pong* ball.

Example 5: A model airplane has mass $m = 1.5\,\mathrm{kg}$ and wing area $A = 0.2\,\mathrm{m}^2$. The plane is kept afloat by air streaming past its wings. Ignore buoyancy due to air. **a)** Calculate the pressure difference between upper and lower surfaces to keep the airplane afloat. **b)** The wings are designed such that air rushes across the upper surface at twice the speed it rushes across the lower surface. Find the speed of air flow over the upper surface so that the airplane does not fall to the ground. Assume $\rho = 1.29\,\mathrm{kg/m^3}$ for the density of air.

Solution: The weight of the plane is $mg = 14.7\,\mathrm{N}$. Then

$$\Delta P = \frac{mg}{A} = \frac{14.7\,\mathrm{N}}{0.2\,\mathrm{m}^2} = 73.5\,\mathrm{Pa}.$$

We write $v_{\mathrm{u}} = v_0$ for the speed of air flow over the upper surface and $v_{\mathrm{l}} = v_0/2$ for the speed of air flow over the lower surface.

$$\Delta P = \frac{1}{2}\rho\left(v_{\mathrm{u}}^2 - v_{\mathrm{l}}^2\right) = \frac{1}{2}\rho\left(v_0^2 - \frac{1}{4}v_0^2\right) = \frac{3}{8}\rho v_0^2$$

so that for the velocity:

$$v_0^2 = \frac{8\Delta P}{3\rho} = \frac{8 \cdot 73.5\,\mathrm{Pa}}{3 \cdot 1.29\,\mathrm{kg/m^3}} = 152\,\frac{\mathrm{m}^2}{\mathrm{s}^2}.$$

Thus, $v_0 = 12.3\,\mathrm{m/s}$.

Discussion: This view of air lift is too simplistic, as airflow across wings is much more complex. Vortices, in particular, appear to play an important role. Also, planes can fly upside down, which would be impossible based on the simple explanation given here.

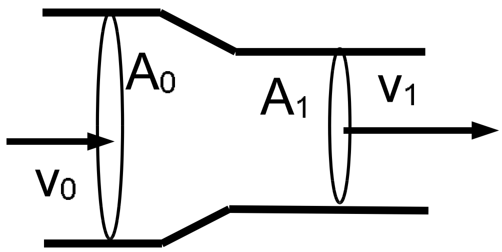
We consider a flow through a pipe with cross-sectional area A . The flow speed is v , which we assume is constant. If x is the coordinate along the pipe, volume element is $\Delta V = A\Delta x$. Since the flow speed is $v = \Delta x/\Delta t$, the *volume flow rate* follows $\Delta V/\Delta = A(\Delta x/\Delta t)$, or

$$\frac{\Delta V}{\Delta t} = Av. \tag{7}$$

The unit of volume flow rate $[\Delta V/\Delta t] = \mathrm{m}^3/\mathrm{s}$. The mass of the fluid inside the volume element is $\Delta m = \rho\Delta V$. We then have for the *mass flow rate* $\Delta m/\Delta t = \rho\Delta V/\Delta t$, or

$$\frac{\Delta m}{\Delta t} = \rho Av, \tag{8}$$

with unit $[\Delta m/\Delta t] = \mathrm{kg/s}$.



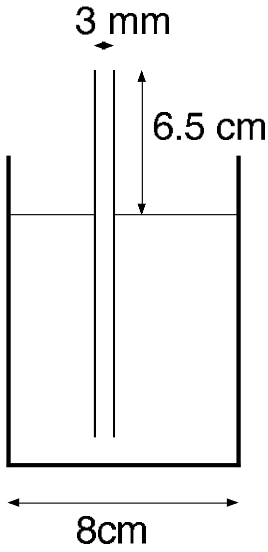
We now consider a pipe with different cross-sectional areas A_0 and $A_1 < A_0$. The incoming mass of fluid must be the same as the outgoing mass, so we have

$$\rho_1 A_1 v_1 = \rho_0 A_0 v_0. \tag{9}$$

When the fluid is incompressible [e.g., for liquids], this simplifies

$$A_1 v_1 = A_0 v_0. \tag{10}$$

Eqs. (9) and (10) are referred to as the *equation of continuity*; they reflect the conservation of matter.



Example 6: A drinking straw 20 cm long and 3.0 mm in diameter stands vertically in a cup of juice 8.0 cm in diameter. A section of the straw 6.5 cm long extends above the juice. A child sucks on the straw and the level of the juice begins to drop at 0.2 cm/s. **a)** What is the speed of the juice inside the straw? **b)** By how much does the pressure in the child's mouth differ from atmospheric pressure?

Solution: We use the equation of continuity $A_1 v_1 = A_2 v_2$ so that

$$v_2 = \frac{A_1}{A_2} v_1 = \frac{\pi(4.0\,\mathrm{cm})^2}{\pi(0.15\,\mathrm{cm})^2} \cdot 0.2 \frac{\mathrm{cm}}{\mathrm{s}} = 142\,\frac{\mathrm{cm}}{\mathrm{s}} = 1.42\,\frac{\mathrm{m}}{\mathrm{s}}.$$

We choose $y = 0$ at the level of juice. The Bernoulli equation reads:

$$P_{\mathrm{atm}} = P_{\mathrm{mouth}} + \rho g y + \frac{\rho}{2} v^2,$$

where P_{mouth} is the pressure in the child's mouth. Thus

$$P_{\mathrm{mouth}} - P_{\mathrm{atm}} = -\rho \cdot \left(gh + \frac{v^2}{2}\right) = -1000\,\frac{\mathrm{kg}}{\mathrm{m}^2}\left(0.065\,\mathrm{m} \cdot 9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} + \frac{(1.42\,\mathrm{m/s})^2}{2}\right) = -1.65\,\mathrm{kPa}.$$

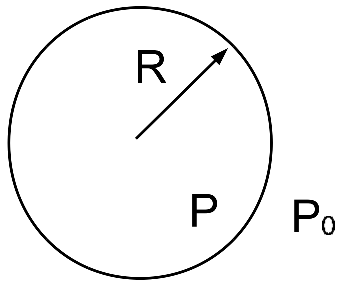
That is, the pressure in the child's mouth is less than the atmospheric pressure.

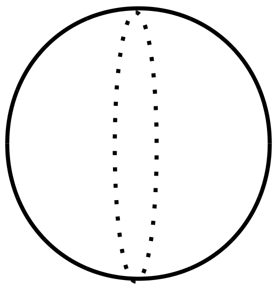
10.3 Surface Tension

The equation $P = -\Delta W / \Delta V$ can be interpreted as the work necessary to "create" volume. Likewise, work is necessary to create surface area ΔA, and we define

$$\sigma = -\frac{\Delta W}{\Delta A}. \tag{11}$$

The quantity σ is referred to as *surface tension* with unit $[\sigma = \mathrm{J/m^2} = \mathrm{N/m}]$. Alternatively, pressure is the force divided by area $P = F/A$ and the surface tension is equal to force divided by length $\sigma = F/L$.





We consider a droplet of radius R. The pressure inside the droplet is P, while the pressure outside is P_0 [e.g., atmospheric pressure]. We assume that the droplet is in mechanical equilibrium so that it neither grows or shrinks. We cut the droplets into two hemispherical shells: the pressure difference produces the force $(P - P_0) \cdot \pi R^2$. This outward force is balanced by the force produced by the surface tension $\sigma \cdot 2\pi R$. We set the two forces equal to each other, and find

$$P - P_0 = \frac{\sigma}{2R}. \tag{12}$$

This is called *Laplace law*. The pressure difference thus increases with decreasing size of the droplet. For water droplets:

R	ΔP [atm]
1 mm	0.0014
0.1 mm	0.0144
1 μm	1.436
10 nm	143.6

We consider two small droplets with radii R_1 and $R_2 < R_1$, respectively: the total energy can be reduced if the small droplet shrinks and the larger droplet grows. This mechanism is responsible for the formation of large rain drops in the atmosphere.

11 Waves

11.1 Transverse and Longitudinal Waves

We are familiar with waves on water surfaces such as lakes and oceans. We observe crests and troughs traveling towards the beach at a constant speed v. The motion of crests and trough is called *wave propagation*. If we take a picture, crests and troughs are observed at equal distances; the distance between troughs [crests] defines the wavelength λ. If we focus on an object [such as a buoy] on the water surface, the object does not travel towards the shore, but rather moves up-and down with a period *T*. Because the displacement is perpendicular to the direction of wave propagation, we say that a surface water wave is a *transverse wave*.

Sound is another familiar example. There is no macroscopic flow of air ['wind'] associated with sound. Air [and liquids] have no stiffness, and sound is associated with the periodic condensation and expansion of air in the direction of wave propagation: sound is a *longitudinal* wave. At a fixed time, regions of compression and expansions are separated by the wavelength λ. On the other hand, at a fixed postion, the air alternatives between condensation and expansion with a period *T*.

The wave speed v, the wave length λ, and the period *T*, of frequency $f = 1/T$, are related

$$v = \frac{\lambda}{T} = f\lambda. \tag{13}$$

Waves propagate energy and momentum; however, no long-range (net) transport of matter is associated with wave phenomena.

Example 1: The speed of surface waves in shallow waters only depends on the depth d of water: $v = \sqrt{gd}$. In water with depth $d = 0.65\,\mathrm{m}$, find the period of the oscillatory motion associated with wave motion of wavelength $\lambda = 1.7\,\mathrm{m}$.

Solution: We have the wave speed:

$$v = \sqrt{9.8\,\frac{\mathrm{m}}{\mathrm{s}^2} \cdot 0.65\,\mathrm{m}} = 2.5\,\frac{\mathrm{m}}{\mathrm{s}}.$$

For $\lambda = 1.7\,\mathrm{m}$, we find the period T:

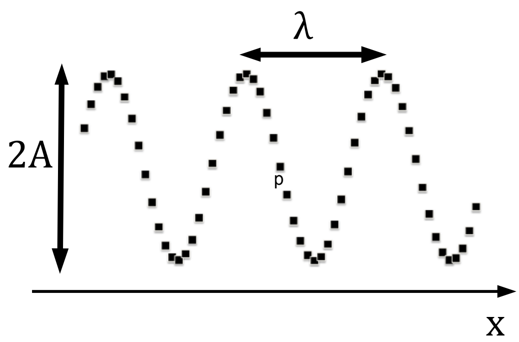
$$T = \frac{\lambda}{v} = \frac{1.7\,\mathrm{m}}{2.5\,\mathrm{m/s}} = 0.68\,\mathrm{s},$$

or frequency $f = 1/(.68\,\mathrm{s}) = 1.5\,\mathrm{Hz}$.

Discussion: The wave speed decreases as the depth decreases, i.e., as the wave moves towards the shore. Because the frequency remains constant, the wavelength decreases when the wave travels towards the shore. If the equation for the wave speed is used for deep ocean woth $d = 4000\,\mathrm{m}$, the wave speed follows $v \simeq 200\,\mathrm{m/s}$. This shows that the propagation of disturbances on the ocean is fast [e.g., during a tsunami].

In the above example, the wave speed is independent of the wavelength or the frequency. This is not the case in general, and the wave speed depends on wavelength $v = v(\lambda)$ or the frequency $v = v(f)$: this dependence is referred to as *dispersion*.

11.2 Wave Speed



We consider a rope [elastic string] and excite it by attaching a vibrator at one end. A transverse wave develops along the wave that is travelling along the x-axis. Each (small) element of the string can be consider to be a harmonic oscillator along the vertical. The elasticity of the rope provides a coupling between oscillators: if a particular oscillator moves up (down), the oscillators in its vicinity also move up (down).

The oscillator at the end with the vibrator undergoes driven harmonic motion. As a result the amplitude A of the oscillator is much larger than the maximum displacement of the vibrator. We can find an expression for the wave speed using dimensional analysis. We replace the the wavelength by the length of the rope $\lambda \sim L$. We assume that the elastic coupling between the neighboring pieces of the rope produces an "effective" spring constant k. If the mass of the spring is m, then the period of the oscillatory motion is $T \sim \sqrt{m/k}$. The wave speed follows $v \sim L\sqrt{k/m} = \sqrt{kL/(m/L)}$. The quantity m/L is the mass per unit length, and we identify kL with the tension along the string $T \sim kL$. Despite the approximate nature of our calculation, we arrive at the correct expression for the wave speed:

$$v = \sqrt{\frac{T}{m/L}}. \tag{14}$$

If the rope is pulled more 'taut,' i.e., when the tension is increased, the wave speed increases.

For sound waves in a gas [longitudinal waves], we consider a volume element $V = L^3$. We write the force in terms of the pressure $F = PL^2$, $v \sim \sqrt{PL^2/(m/L)} = \sqrt{P/(m/L^3)}$. The expression in the denominator is the density so that $v \sim \sqrt{P/\rho}$. In the next chapter, we see that the density of a gas is proportional to the pressure and inversely proportional to the (absolute) temperature T, $\rho = m/V = MP/kT$, where is the mass of a molecule and k $[k = 1.38 \times 10^{-21}\ \mathrm{J/K}]$ is the Boltzmann constant, $v \sim \sqrt{kT/M}$. This yields:

$$v = \sqrt{\frac{\gamma kT}{M}}, \tag{15}$$

where the adiabatic exponent $\gamma = 5/3$ for diatomic gas, such as air. For most applications, the sound speed can be assumed constant $v = 345\,\mathrm{m/s}$. The sound speed depends on the phase of the material [solid, liquid, or gas],

$$v_{\mathrm{gas}} < v_{\mathrm{liquid}} < v_{\mathrm{solid}}. \tag{16}$$

Typical values are $v \sim 5000\,\mathrm{m/s}$ for sound in solids, $v \sim 1000\,\mathrm{m/s}$ for liquids, and $v \sim 500\,\mathrm{m/s}$ for gases.

Example 2: A guitar string has length $L = 0.56\,\mathrm{m}$ and mass $m = 2.4\,\mathrm{g}$. Find the tension of a guitar string so that the speed of the transverse wave is equal to to speed of sound in air.

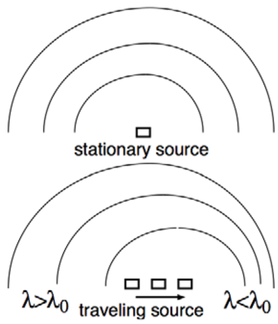
Solution: We solve for the tension along the string:

$$F = \frac{m}{L}v^2 = \frac{2.4 \times 10^{-3}\,\mathrm{kg}}{0.56\,\mathrm{m}} \left(345\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 = 510\,\mathrm{N}.$$

Discussion: When a musician "tunes" the guitar, the wave speed along the string changes. Because the wavelength is fixed, the frequency of the sound changes as a result of the change in the wave speed.

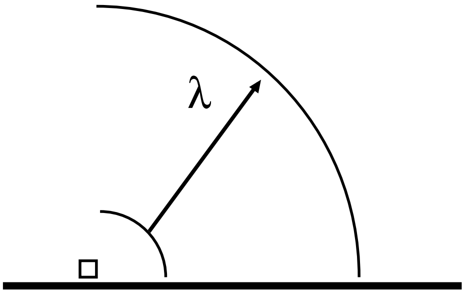
11.3 Doppler Effect

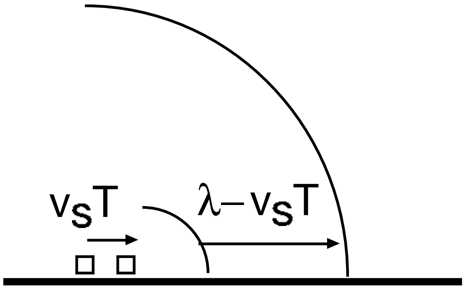
We only consider the case when source and observer are moving along a straight line. We distinguish between the case (1) when the observer is stationary and the sourse is moving, and (2) when the observer is moving and the source is stationary.



The source is producing a wave crest every period $t_n = nT$ with $n = 0, 1, 2, 3 \ldots$. At some time *t*, the n -th crest has coordinate $\pm x_n$. Because each crest travels at the wave speed v, the distance between consecutive crests is $\Delta x = x_{n+1} - x_n = \lambda$.

If the source travels with speed v_s, the distance between crests is decreased when the source moves towards a stationary observer $\lambda' = \lambda - v_s T < \lambda$ and is increased when the source moves away from the from observer $\lambda' = \lambda + v_s T > \lambda$.





We note that the speed of the wave crest v is independent of the speed of the source v_s. Because the distance between crests is $\lambda' = \lambda \pm v_s T$, the observer notices crests at the rate (frequency) v/λ', or

$$f_o = \frac{v}{\lambda'} = \frac{v}{\lambda \pm v_s T} = \frac{v/\lambda}{1 \pm v_s T/\lambda}.$$

Since $v/\lambda = f_s$ is the frequency of the stationary source and $\lambda/T = v$ is the wave speed, we find

$$f_o = \frac{f_s}{1 \pm v_s/v}, \tag{17}$$

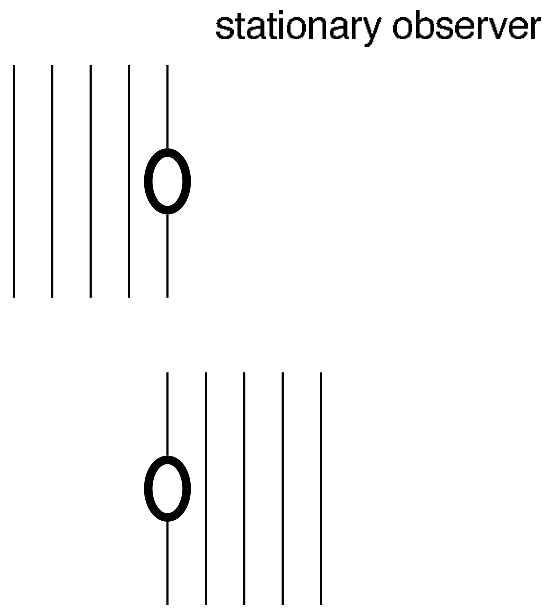
where f_s is the frequency of stationary source. Here the "+" ("-") sign applies when the source moves towards (away from) the stationary observer.

Example 3: A locomotive plays a horn with frequency $f_s = 1500\,\mathrm{Hz}$. What is the frequency heard by a passenger waiting at a train station when the locomotive enters (leaves) the station at the speed $v_s = 40\,\mathrm{m/s}$.

Solution: We have

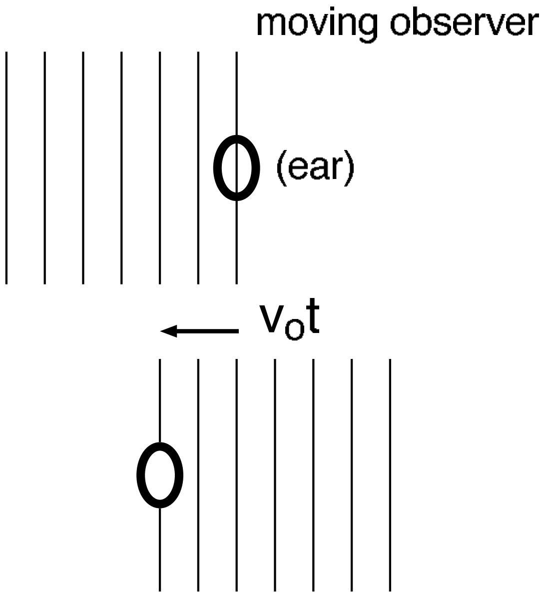
$$f_o = \frac{1500\,\mathrm{Hz}}{1 \pm (40\,\mathrm{m/s})/(345\,\mathrm{m/s})} = \begin{cases} 1344\,\mathrm{Hz} & \text{locomotive entering station} \\ 1697\,\mathrm{Hz} & \text{locomotive leaving station} \end{cases}$$

Discussion: This frequency change is easily detected by the human ear.



When both source and observer [represented by an 'ear'] are stationary, the number of intercepted wavefront by observer in a time interval t is vt/λ . The frequency of the observed wave is this number divided by the time t :

$$f_o = \frac{vt/\lambda}{t} = \frac{v}{\lambda}, \quad v_o = 0.$$



When the observer travels towards the source with the speed v_o (here towards the *left*), the speed of the wave crests relative to the observer is $v + v_o$. That is, the number of crests "collected" by the ear in a time t is $(vt + v_o t)/\lambda$. Thus, the observed frequency detected by the ear is

$$f_o = \frac{(vt + v_o t)/\lambda}{t} = \frac{v + v_o}{\lambda} = \frac{v}{\lambda}\left(1 + \frac{v_o}{v}\right).$$

Since $f_s = v/\lambda$ is the frequency of the source, the frequency when the source is stationary and the observer is moving:

$$f_o = f_s\left(1 \pm \frac{v_o}{v}\right), \tag{18}$$

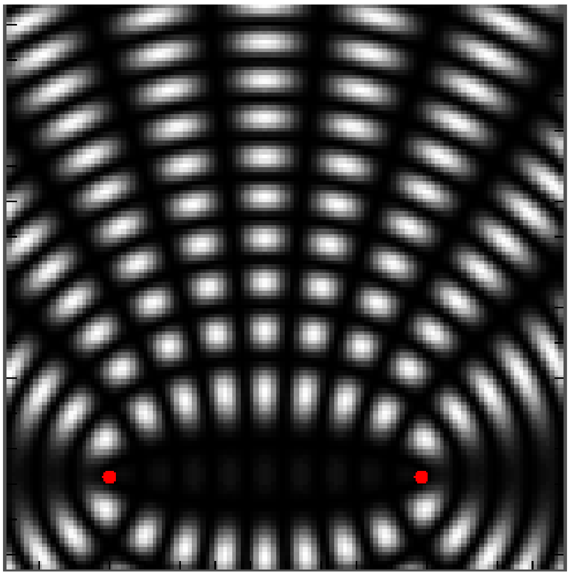
where f_s is the frequency of the source. Here, the "+" ("-") sign applies when the observer moves towards (away from) the stationary source.

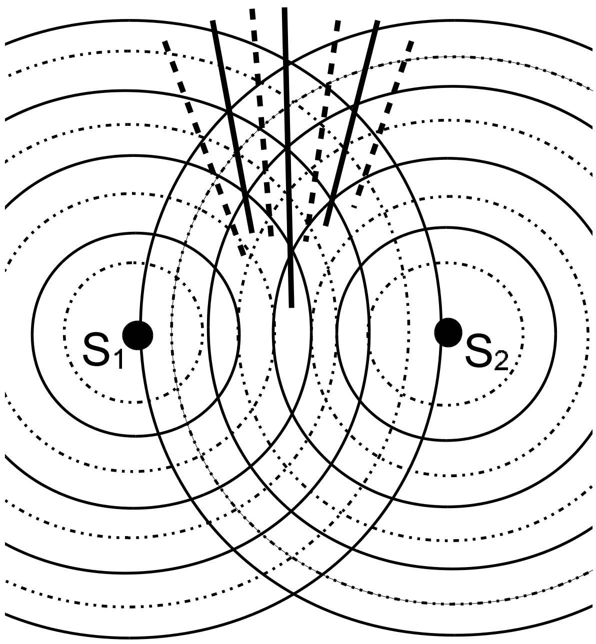
Example 4: A bungee jumper falls towards the bottom of the valley. A horn with frequency $f_s = 690\,\mathrm{Hz}$ blares from the bottom of the valley. What is the frequency observed by the bungee jumper when she has fallen the height $h = 13\,\mathrm{m}$.

Solution: We find the speed of the jumper from free fall:
$v_s = \sqrt{2gh} = \sqrt{2 \cdot 9.8\,\mathrm{m/s^2} \cdot 13.0\,\mathrm{m}} = 16.0\,\mathrm{m/s}$. We then have for the frequency heard by the jumper:

$$f_o = 690\,\mathrm{Hz} \cdot \left(1 + \frac{16\,\mathrm{m/s}}{343\,\mathrm{m/s}}\right) = 722\,\mathrm{Hz}.$$

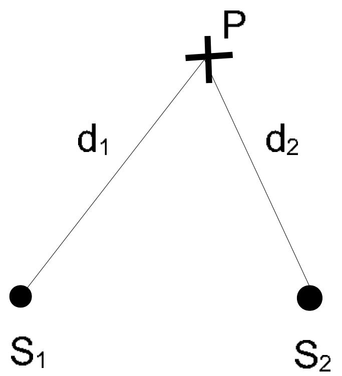
11.4 Interference





We assume that two *coherent* sources S_1 and S_2 (indicated by the two red dots] produce waves in the same region. Coherent means that successive crests [solid] and troughs [dotted] of the two sources are produced at the same time. The addition of two waves follows the *superposition principle*: When two or more waves are present simultaneously at the same location, the resultant disturbance is the sum of the disturbances from the individual waves; that is, we add the amplitudes and not the intensities from different sources.

In the forward direction, a crest (trough) from S_1 'meets' a crest (trough) from S_2. Along this (solid) center line, a wave with amplitde $2A$ develops: we say that the waves add *constructively*. The direction of the center line define the zeroth-order maximum. On either side of this line, two dotted lines connect dots, where a crest (trough) from S_1 meets a trough (crest) from S_2: we say that the two waves add *destructively*, i.e., they cancel out each other. The directions of these two lines define the first-order minimum. We then indicate with solid lines the points, where a crest (trough) from S_1 meets a crest (trough) from S_2. These lines define the first order maximum.

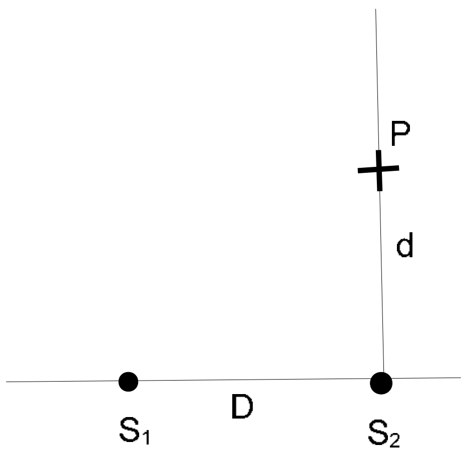


The condition for a maximum depends on the distances d_1 and d_2 between the two sources S_1 and S_2 and the observation point P:

$$|d_1 - d_2| = n\,\lambda, \qquad n = 0, 1, 2, ... \tag{19}$$

so that $n = 0$ for the zeroth-order maximum defines the center line. For the minimum, we have

$$|d_1 - d_2| = \left(n + \frac{1}{2}\right)\lambda, \qquad n = 0, 1, 2, ... \tag{20}$$



Example 5: Two speakers vibrate in phase and produce the sound with frequency $f = 440\,\mathrm{Hz}$ [musical tone "A"]. The speakers are separated by the distance $D = 23\,\mathrm{cm}$. Find the shortest perpendicular distance from the speaker S_2, where the sound from the two speakers cancels. What is the distance to the next maximum?

Solution: We find the wave length of the sound $\lambda = (345\,\mathrm{m/s})/(440\,\mathrm{m}) \quad = 0.78\,\mathrm{m}$. Since $n = 0$, distance d_1 follows from Pythagorean theorem: $(\lambda/2)^2 = D^2 + d_1^2$ so that

$$d_1 = \sqrt{\left(\frac{0.78\,\mathrm{m}}{2}\right)^2 - (0.23\,\mathrm{m})^2} = 0.32\,\mathrm{m}.$$

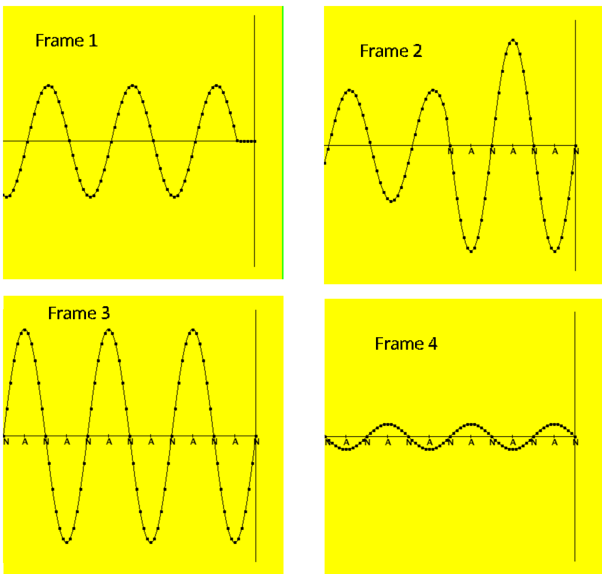
We set $n = 1$ and find the distance to the first maximum:

$$d_2 = \sqrt{(0.78\,\mathrm{m})^2 - (0.23\,\mathrm{m})^2} = 0.75\,\mathrm{m}.$$

The distance between maximum and minimum is $\Delta d = d_2 - d_1 = 0.44\,\mathrm{m}$.

Interference is a characteristic feature for all waves, independent of the nature of the wave, and explains several wave phenomena: (1) standing waves that are relevant for waves along a string and the sound waves and (2) diffraction.

11.5 Standing Waves



A special case of interference applies when a wave interferes with a reflected wave: the two waves are automatically vibrating in phase. We consider the case when the original wave travels to the right [Frame 1]. The reflected wave is generated at the wall and travels to the left. The superposition of the original and the relected yields the pattern shown in Frame 2. This pattern changes with time [transient behavior]. After a long time, $t \to \infty$, the pattern of crests and troughs is time-independent [stationary]. There are locations, where the (vertical) displacement is always zero; these locations are called *nodes*. In the middle between two nodes is a location with maxium (vertical) displace- ment. The locations of maximum displace- ments are referred to as *antinodes* [Frame 3]. Each part of the string moves in 'unison;' that is, each antinode reaches the maxium displacement at the same time, while at other times, the displacements are less than the maximum value [Frame 4]. This resulting wave is referred to as a *standing wave*, and is independent of the nature of the wave [transverse or longitudinal] and thus applies to a wave along a string [as shown], the sound waves in an organ pipe, etc.

The time between successive maximum displacements is the period, or the inverse of the frequency of the wave, $T = 1/f$. The distance between neighboring nodes is half a wavelength $\lambda/2$. If the two ends of a rope are fixed, or the two ends of a pipe are closed, the wave has nodes at the two ends so that $L = n\lambda/2$, for some integer $n = 1, 2, 3, \ldots$. This implies that the length of the rope or the organ pipe determines the frequencies of the produced sound:

$$f_n = \frac{v}{\lambda_n} = \frac{v}{2L}n, \qquad n = 1, 2, 3... \tag{21}$$

The different cases are called the n -th harmonics, i.e., first, second, etc harmonic. In a music context, the second (third) harmonic is called the first (second) overtone.

If one end is fixed and the other end swings freely [e.g., one end of the organ pipe is open, while the other side is closed] then there is a node at the fixed end and an antinode at the open end. We get for the length $L = n\lambda/4$ for $n = 1, 3, 5, \ldots$ so that for the possible frequencies,

$$f_n = n\left(\frac{v}{4L}\right), \quad n = 1, 3, 5, \ldots$$

That is, there are no even harmonics in 'asymmetric' situations.

Example 6: The G string on a violin is 30 cm long. When played without *fingering*, it vibrates at a frequency of 196 Hz. **a)** What is the wave speed along the string? **b)** The mass of of the G-string is 1.2 g; Find the tension in the G-string. **c)** Another note on the C-major scale is D at 294 Hz. How far from the end of the string must a finger be placed to play this note?

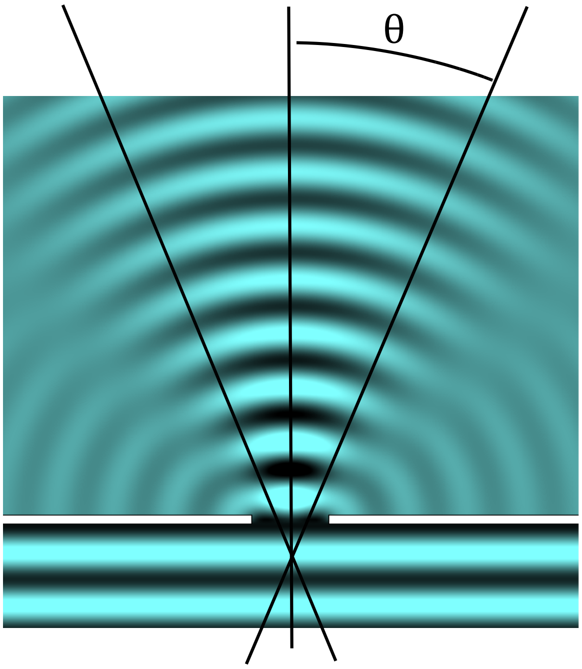
Solution: Because $f_1 = 196\,\mathrm{Hz}$ is the first harmonic, we have $L = 0.3\,\mathrm{m} = \lambda/2$ so that $\lambda = 0.6\,\mathrm{m}$. We thus have the wave speed: $v = \lambda f = 0.6\,\mathrm{m} \cdot 196\,\mathrm{Hz} = 117.6\,\mathrm{m/s}$. The tension in the string follows from the wave speed $v^2 = T\,(m/L)$ so that $T = (m/L) \cdot v^2 = 55.3\,\mathrm{N}$. Since $f_1' = 294\,\mathrm{Hz}$, we get the new wavelength:

$$\lambda' = \frac{v}{f_1'} = \frac{117.6\,\mathrm{m/s}}{294\,\mathrm{Hz}} = 0.4\,\mathrm{m}.$$

Since this is the first harmonic for the shortened string, we find $L' = \lambda'/2 = 0.2\,\mathrm{m}$. That is, the finger must be placed at the distance $D = L - L' = 10\,\mathrm{cm}$ from the end of the string.

If a musical instrument, say a piano, plays the tone *A* [with frequency 440 Hz], the vibrating string generates not only the first harmonic $f_1 = 440\,\mathrm{Hz}$ but also the higher harmonics [overtones] $f_2 = 2f_1 = 880\,\mathrm{Hz}$, $f_3 = 3f_1 = 1320\,\mathrm{Hz}$, etc, each with a different intensity. We say that the string generates a 'complex' sound characterized by a frequency spectrum; the difference in the spectrum is used to distinguish the sound from different instruments: piano, horn, etc.

11.6 Diffraction



We assume that a plane wave travels towards a wall with an opening of size D. Behind the wall, we observe wave within a cone of angle θ around the forward direction. At greater angles, the interference leads to cancellation of the wave. This phenomenonis called *diffraction*: loosely speaking, diffraction is the bending of waves around a corner. Diffraction originates from the interference of outgoing waves originating from the opening. It makes it possible to hear people from room to room, even if we can't see them [also reflection on walls, floor, and ceiling plays a role].

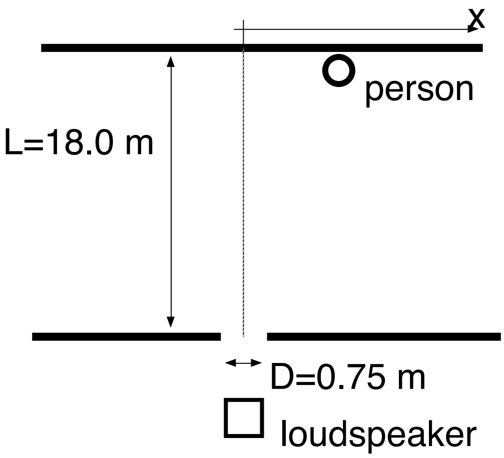
We use dimensional analysis to find the angle of diffraction θ. The angle is dimensionless $[\theta] = 1$, and depends on the ratio of two length scales: the wavelength λ and the diameter of the opening D. The diffraction phenomenon disappears when either the opening becomes very large or the wavelength becomes very small,

$$\sin\theta = \frac{\lambda}{D} \quad \text{[rectangular opening]}. \tag{22}$$

When the opening is a circle, the above expression must be slightly modified:

$$\sin\theta = 1.22\,\frac{\lambda}{D}, \qquad \text{[circular opening]}. \tag{23}$$

The factor 1.22 cannot be derived with the tools of undergraduate physics [and mathematics].



Example 7: A loudspeaker is behind a rectangular opening that is 0.75 m wide. The opening is 18.0 m from a wall, where a person listens to the sound. The loudspeaker produces sounds with different frequencies. Use $v = 343\,\mathrm{m/s}$ for the speed of sound in air. **a)** The loudspeaker produces sound with frequency $f = 6.0\,\mathrm{kHz}$. Find the maximum distance from the center, where she can clearly hear the sound. **b)** What is the highest frequency that the person can hear when she is standing at a distance $x' = 2.3\,\mathrm{m}$ from the center?

Solution: We have the wavelength of the sound: $\lambda = v/f = (343\,\mathrm{m/s})/(6.0 \times 10^3\,\mathrm{s}^{-1}) = 0.057\,\mathrm{m}$. The opening has width $D = 0.75\,\mathrm{m}$.

This gives the diffraction angle

$$\sin\theta = \frac{\lambda}{D} = \frac{0.057\,\mathrm{m}}{0.75\,\mathrm{m}} = 0.076, \quad \longrightarrow \quad \theta = 4.4°.$$

Because the person is at a distance $L = 18.0\,\mathrm{m}$ from the opening, the distance of the person from the center

$$x = L\tan\theta = 18.0\,\mathrm{m} \cdot \tan 4.4° = 1.38\,\mathrm{m}.$$

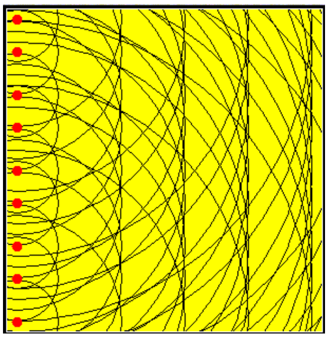
We have $\tan\theta = x/L = (2.3\,\mathrm{m})/(18.0\,\mathrm{m}) = 0.128$ so that $\theta = 7.3°$. This gives the shortest wavelength:

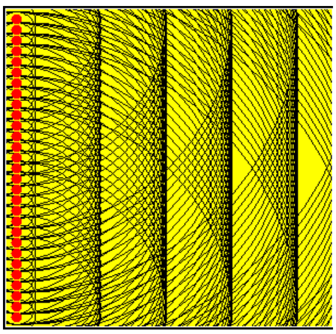
$$\lambda = D\sin\theta = 0.75\,\mathrm{m} \cdot \sin 7.3° = 0.095\,\mathrm{m}.$$

The frequency then follows

$$f = \frac{v}{\lambda} = \frac{343\,\mathrm{m}}{0.095\,\mathrm{m}} = 3.6\,\mathrm{kHz}.$$

Discussion: The aperture dimension 0.75 m is much larger than typical speaker cones.





A theoretical derivation of diffraction is based on *Huygens' principle* for wave propagation. All points of a wave front can be considered as sources of outgoing spherical waves [idicated by the red dots]. These spherical waves interfere constructively and destructively at different points in the plane. The *envelope* of the spherical waves then forms the next wave front, and the process repeats itself. The new wave front becomes clearer when the number of sources of outgoing spherical waves is increased. This description is exact in the limit as the number of sources becomes infinitely large.

12 Thermal Physics

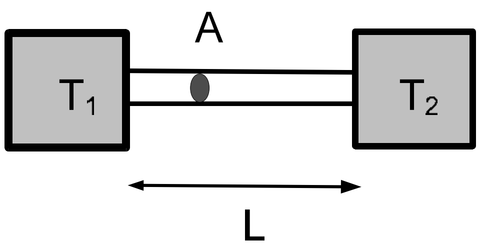
12.1 Temperature and Heat

While a formal definition of temperature is outside the scope of introductory physics, "temperature is the thing that's the same for two objects, after they've been in contact long enough" [D.V. Schroeder, *Thermal Physics* (Addison-Wesley, San Francisco, 2000)]. When the two bodies are in contact, they will exchange energy with each other. Temperature is the quantity that describes such a *spontaneous* flow of energy [heat] between objects at different temperatures. Energy flows from the hotter object to the colder object. Two ojects have the same temperature when there is no net flow of heat between them. The *zeroth law of thermodynamics* states that two bodies, each in thermal equilibrium with a third, are also in thermal equilibrium with each other.

Properties of water [$\mathrm{H_2O}$] are used to define the *Celsius* temperature scale. Liquid water and ice coexist at $T_c = 0°\mathrm{C}$ and liquid water and vapor coexist at $T_c = 100°\mathrm{C}$. The absolute temperature scale is

$$T = T_c + 273.15. \tag{24}$$

The unit of temperature is $[T] = \mathrm{K}$ (Kelvin).



Heat flow: Heat flow is the rate of energy transfer or Q/t . Two objects are kept at temperatures T_0 and T_1, respectively. The two bodies are connected by a heat conductor [e.g., a metal strip] of length *L* and cross- sectional area *A*. The flow of heat depends on the "steepness" the temperature change $\Delta T/L$ [temperature *gradient*]. The heat flow is proportional to the cross-sectional area *A*:

$$\frac{Q}{t} = k \cdot A \cdot \frac{\Delta T}{L}. \tag{25}$$

Here, k is the thermal conductivity with unit $[k] = \mathrm{W/(K\,m)}$, and must in general be measured experimentally. Good heat conductors are also good conductors of electricity. Thus metals are good heat conductors, whereas insulators [wood and (most) plastics] are bad conductors. Humans are sensitive to heat flow, and not the temperature. After a cold winter night, the wood on the porch 'feels' warmer than the metal railing, although they are at the same temperature. The different "feel' is due to difference in thermal conductivities of the two materials [greater for metal than for wood].

Heat can also be transferred through (blackbody) radiation,

$$\frac{Q}{t} = e\sigma A T^4,$$

where *A* is the surface area, e is the emissivity $[e < 1]$ and σ is the Stefan-Boltzmann constant $\sigma = 5.67 \times 10^{-8}\mathrm{W/(m^2 \cdot K^4)}$.

Example 1: The temperature of skin is $T_{\text{skin}} = 34°\mathrm{C}$ [somewhat less than the body temperature $[T_{\text{body}} = 37°\mathrm{C}]$ and the ambient temperature is $T_{\text{room}} = 24°\mathrm{C}$. The surface area of an adult is $A \simeq 1.85\,\mathrm{m}^2$. Calculate the net heat loss of the person.

Solution: Applied to the human body: skin temperature $T_{\text{skin}} = 34°\text{C} = 307\,\text{K}$. We assume that the room temperature is at a "comfortable" $T_{\text{room}} = 24°\text{C} = 297\,\text{K}$. The surface area of the human body is $A \simeq 1.85\,\text{m}^2$. We assume $\epsilon \simeq 1$. Then

$$\begin{aligned}\left(\frac{Q}{t}\right)_{\text{net}} &= \left(\frac{Q}{t}\right)_{\text{gain}} - \left(\frac{Q}{t}\right)_{\text{loss}} \\ &= 1 \cdot 5.67 \times 10^{-8} \frac{\text{W}}{\text{m}^2\text{K}^4} \cdot 1.85\text{m}^2 \cdot \left[(297\,\text{K})^4 - (307\,\text{K})^4\right] \\ &= 816\,\text{W} - 932\,\text{W} \simeq -100\,\text{W}.\end{aligned}$$

That is, this result agrees with the metabolic rate of a person.

Discussion: Four main sources of heat loss in humans have been identified: (1) radiation (black -body radiation): see above 54-60% (2) convection and conduction of air from body $\sim 25\%$ (3) evaporation of sweat $\sim 7\%$ (4) evporation of water from breathing $\sim 14\%$. [source: I.P. Herman, *Physics of the Human Body* (Springer, New York, 2007)]

A temperature increase generally increases the size of an object. For a long strip, the *fractional* increase of length is proportional to the temperature change:

$$\frac{\Delta L}{L_0} = \alpha \Delta T, \tag{26}$$

while we have the *fractional* volume increase for a bulk object:

$$\frac{\Delta V}{V} = \gamma \Delta T. \tag{27}$$

Here the coefficients of linear α and volume expansion γ are related, $\gamma = 3\alpha$. The units are $[\alpha] = [\gamma] = (\ °\text{C})^{-1}$.

Example 2: Calculate the linear variation of a 50-m long steel T-beam due the seasonal temperature change. Use $v(t/2) = v_0 + a(t/2)$ for steel.

Solution: We have $T_{\text{low}} = -20°\text{C}$ and $T_{\text{high}} = +40°\text{C}$ so that $\Delta T = 60°\text{C}$. Then

$$\frac{\Delta L}{L_0} = 12 \times 10^{-6} \frac{1}{°\text{C}} \cdot 60°\text{C} = 7.2 \times 10^{-4} \quad (= 0.072\%),$$

so that the fractional change is 0.072%. For $L = 50\,\text{m}$, the length change is $\Delta L = 3.6\,\text{cm}$.

Discussion: Such a change in the length can produce significant problems for structural engineers who incorporate expansion joints in bridges and other structures.

12.2 Calorimetry

Since heat flows spontateneously from object with high T to an object with low T, the temperature of a body changes when heat is added or removed. The heat Q is proportional to the size of the object [mass m] and also depends on the type of material,

$$Q = mc\Delta T, \tag{28}$$

where c is the specific heat. Since heat is a form of energy [just like kinetic and potential energy], the unit of the specific heat is $c = \mathrm{J}/(\mathrm{kg}\ {}^\circ\mathrm{C})$. Heat must be added or removed during a phase change [gas-liquid (evporation or condensation), liquid-solid (melting or freezing), or gas-solid (sublimation)]. Because the temperature is constant during a phase change,

$$Q = mL,$$

where the latent heat L has the unit $[L] = \mathrm{J}/\mathrm{kg}$. The specific and latent heat for materials are measured experimentally. The specific heats are different for the different phases of the same substance and likewise, the latent heats are different for different phase transformations. Furthermore, the specific and latent heats are different for different substances.

Example 3: The best way to make *hot chocolate* is by putting steam into chilled milk. An Espresso machine produces vapor at $100°\mathrm{C}$ and pressure $P = 9.0 \times 10^5\,\mathrm{Pa}$ [or 9 atm]. The barista puts $m_{\text{steam}} = 50.4\,\mathrm{g}$ $[V = 9.6 \times 10^{-3}\,\mathrm{m}^3$ or 9.6 L] of steam into the chilled milk. A 12-ounce cup of milk contains 0.36 kg of milk at the temperature $6°\mathrm{C}$. Find the temperature of the *Hot Chocolate*! Ignore the specific heat of the cup. Useful data: Specific heat of water $c_{\text{water}} = 4,186\,\mathrm{J/(kg \cdot^\circ C)}$, Specific heat of milk $c_{\text{milk}} = 3,890\,\mathrm{J/(kg \cdot^\circ C)}$, and latent heat of vaporization of water $L_{\text{water}} = 22.6 \times 10^5\,\mathrm{J/kg}$.

Solution:We have for the heat given off by the vapor [and hot water] and the absorbed heat:

$$Q^{\uparrow} = m_{\text{steam}} L_{\text{water}} + m_{\text{steam}} c_{\text{water}} (100° - T_f)$$

$$Q^{\downarrow} = m_{\text{milk}} c_{\text{milk}} (T_f - 6°\mathrm{C})$$

Now set $Q^{\uparrow} = Q^{\downarrow}$ and solve for the final temperature T_f:

$$T_f = \frac{m_{\text{steam}}(L_{\text{water}} + c_{\text{water}} 100°) + m_{\text{milk}} c_{\text{milk}} 6°\mathrm{C}}{m_{\text{milk}} c_{\text{milk}} + m_{\text{steam}} c_{\text{water}}} = \frac{134.5\,\mathrm{kJ} + 8.4\,\mathrm{kJ}}{1.40\,\mathrm{kJ/°C} + 0.2\,\mathrm{kJ/°C}} = 89°\mathrm{C}$$

That's about $190°\mathrm{F}$.

Example 4: The composition [i.e., the fraction of pulp] of apples [or any other fruit or vegetable] is measured using calorimetry. Hot water [with mass $m_W = 0.11\,\mathrm{kg}$ in a aluminum beaker [with mass $m_{\text{Al}} = 0.05\,\mathrm{kg}$] at the initial temperature $T_i = 63°\mathrm{C}$. Chipped apples are chilled at $0°\mathrm{C}$ by placing them inside [or close to that!]. The mass of the chipped apple is $M = 0.05\,\mathrm{kg}$. The final temperature is $T_f = 46°\mathrm{C}$. Assume that the apple consists of water and cellulose. How much of the apple is water? Useful data: Specific heat of water $c_{\text{water}} = 4186\,\mathrm{J/(kg\ \ °C)}$, aluminum $c_{\text{Al}} = 900\,\mathrm{J/(kg\ \ °C)}$, and cellulose [pulp] $c_{\text{cellulose}} = 1400\,\mathrm{J/(kg\ \ °C)}$.

Solution The heat given off is by the hot water and the aluminum beaker:

$$Q^{\uparrow} = m_W c_W (T_i - T_f) + m_{\text{Al}} c_{\text{Al}} (T_i - T_f) = \left[460.5\,\frac{\mathrm{J}}{°\mathrm{C}} + 45.0\frac{\mathrm{J}}{°\mathrm{C}}\right] \cdot 17°\mathrm{C} = 8594\,\mathrm{J}$$

The apple absorbs heat:

$$Q^{\downarrow} = M c_{\text{apple}} (T_f - 0°\mathrm{C}), \tag{29}$$

where c_{apple} is the unknown specific of the apple. We assume that we don't have any heat loss so that $Q^{\uparrow} = Q^{\downarrow}$, and solve for the specific heat of the apple,

$$c_{\text{apple}} = \frac{9594\,\text{J}}{0.05\,\text{kg} \cdot 46°\text{C}} = 3736\,\frac{\text{J}}{\text{kg}\ \ °\text{C}}.$$

Then for the mass of apple: $M = M_{\text{w}} + M_{\text{cellulose}}$. We introduce the (mass) fraction of cellulose:

$$r = \frac{M_{\text{cellulose}}}{M}.$$

We then have for the specific heat of apple in terms of the ratio r and the known specific heat: $c_{\text{apple}} = c_W(1-r) + c_{\text{cellulose}}r$. Now solve for the ratio:

$$r = \frac{c_W - c_{\text{apple}}}{c_W - c_{\text{cellulose}}} = \frac{4186\,\text{J}/(\text{kg}°\text{C}) - 3736\,\text{J}/(\text{kg}°\text{C})}{4186\,\text{J}/(\text{kg}°\text{C}) - 1400\,\text{J}/(\text{kg}°\text{C})} = 0.16.$$

That is, 16% of the apple is cellulose and the remainder [84%] is water.

Discussion: In agreement with published data [http://www.ca.uky.edu/enri/pubs/enri129.pdf – retrieved on Aug. 26, 2012]. The (more standard) determination of the percentage of water in fruits and vegetables is done by measuring the average density of apple. The latter is most easily done by measuring the buoyancy of an apple in air and water when it is fully submerged.

12.3 Ideal Gas Law

The ideal gas law connects the pressure, volume, and temperature: (1) for a fixed volume, a higher temperature leads to an increase in pressure, $P \sim T$, (2) at constant pressure, a higher temperature leads to an increase volume, $V \sim T$, and (3) at constant temperature, a higher pressure is associated with a smaller volume, $P \sim 1/V$. This can be summarized as

$$\frac{PV}{T} = \text{const}, \tag{30}$$

where T is the absolute temperature [measured in Kelvin]. This law assume that the amount of gas [i.e., the mass] is kept constant.

Example 5: A glass column is filled with air. At room temperature, the pressue is 2.5-times the atmospheric pressure. What is the pressure inside the glass column when it immersed in boiling water?

Solution: We have the absolute temperatures $T_0 = 298\,\text{K}$ and $T_f = 373\,\text{K}$. Since the volume is constant $V_0 = V_1 = V$, $P_0V/T_0 = P_1V/T$ so that

$$P_1 = \frac{T_1}{T_0}P_0 = \frac{373\,\text{K}}{298\,\text{K}}\,2.5\,\text{atm} = 3.1\,\text{atm}.$$

In an ideal gas, molecules are moving in random directions aat random speeds inside a container. The molecules bounce off the walls. During a collision, the wall of the container exerts a force on a molecule; thus, the molecule exerts a (reaction) force on the container wall: the summation of all forces from all collisions produces the macroscopic pressure.

A simplified derivation starts from a cubic container with volume $V = L^3$; we choose a coordinate system aligned with the cube. We assume that the molecule travels with speed v along the x -coordinate. The collision with the wall gives the change in momentum $\Delta p = (-mv_x) - m(v_x) = -2mv_x$. Since the time between consecutive collisons is $t = 2L/v$, the (average) force by one molecule follows $F_{\text{ave}} = (-2mv_x)/(2L/v_x) = -mv_x^2/L$. The particle moves in all directions so that $v_x^2 = \langle v^2 \rangle /3$, where we introduced the mean square value $\langle v^2 \rangle$. Thus for the force due to N molecules $F = Nm \langle v^2 \rangle /3L$ so that for the pressure $P = F/L^2 = Nm \langle v^2 \rangle /3V$. We find

$$PV = \frac{N}{3} mv_{\text{rms}}^2 = \frac{2}{3} N \left(\frac{1}{2} mv_{\text{rms}}^2 \right) = \frac{2}{3} N \langle KE \rangle , \tag{31}$$

where we introduced the average kinetic energy of the molecule $\langle KE \rangle = m \langle v^2 \rangle /2$. Comparison with ideal gas law $PV/T = \text{const}$ yields

$$\langle KE \rangle = \frac{m}{2} v_{\text{rms}}^2 = \frac{3}{2} kT, \tag{32}$$

where $k = 1.381 \times 10^{-23}$ J/K is the Boltzmann constant.

The number of molecules in a gas is enormous $N >> 1$. Avogardo's number is used to quantify large numbers,

$$N_A = 6.022 \times 10^{23}. \tag{33}$$

The number of *moles* is then given by

$$n = \frac{N}{N_A}. \tag{34}$$

The product of the molecular mass and Avogadro's number is the molar mass $M = N_A m$. For helium $M_{\rm He} \simeq 4\,{\rm g}$ and air [80% $\rm N_2$ and 20% $\rm O_2$]; $M_{\rm air} = 28.8\,{\rm g}$. Avogadro's number and the Boltzmann constant are the connection between microscopic and macroscopic quantities. The unit for masses of molecules is the atomic mass unit:

$$1\,{\rm u} = \frac{1\,{\rm g}}{N_A} = 1.66 \times 10^{-27}\,{\rm kg}. \tag{35}$$

Example 6: Calculate the average speed of a molecule of Argon gas at room temperature [here, atom and molecule mean the same].

Solution: The mass of an Argon atom is $m = 39.95\,{\rm u}$, so that

$$v_{\rm rms} = \sqrt{\langle v^2 \rangle} = \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \cdot 1.38 \times 10^{-23}\,{\rm J/K} \cdot 293\,{\rm K}}{39.95 \times 1.66 \times 10^{-27}\,{\rm kg}}} = 428\,\frac{\rm m}{\rm s}.$$

Discussion: This root-mean-square speed is very close to the speed of sound for the gas. We get for the energy:

$$E_{\rm thermal} = \frac{3}{2}kT = \frac{3}{2} \cdot 1.38 \times 10^{-23}\,\frac{\rm J}{\rm K} \cdot 300\,{\rm K} = 6.2 \times 10^{-21}\,{\rm J}.$$

The convenient unit for molecular energies is the *electron Volt* : $1\,{\rm eV} = 1.6 \times 10^{-19}\,{\rm J}$. Then

$$E_{\rm thermal} \simeq 40\,{\rm meV}.$$

The latent heat of vaporization for water is $L_{\rm vapor} = 22.6 \times 10^5\,{\rm J/kg}$. The number of moles: $n = 1\,{\rm kg}/(18 \times 10^{-3}\,{\rm kg/mol}) = 56\,{\rm mol}$. We write $L = N\epsilon$, so that for the energy per molecules:

$$\epsilon = \frac{22.5 \times 10^5\,{\rm J}}{56 \cdot 6.02 \times 10^{23}} \simeq 6.6 \times 10^{-20}\,{\rm J} = 0.4\,{\rm eV}.$$

The ideal gas law can then be written $PV = (2/3)\, nN_A\, (3kT/2)$ or

$$PV = nRT, \tag{36}$$

where $R = kN_A = 8.31\,\mathrm{J/(mol \cdot K)}$ is the gas constant.

Example 7: Calculate the molar volume at *standard conditions*.

Solution: Standard conditions refers to temperature $T = 0\ °\mathrm{C} = 273\,\mathrm{K}$ and atmospheric pressure $P = 1.013 \times 10^5\,\mathrm{Pa}$. Since $n = 1\,\mathrm{mol}$:

$$V = \frac{nRT}{P} = \frac{1.0\,\mathrm{mol} \cdot 8.31\,\mathrm{J/(mol\,K)} \cdot 273\,\mathrm{K}}{1.013 \times 10^5\,\mathrm{Pa}} = 0.0224 \times 10^{-2}\,\mathrm{m}^3 = 22.4\,\mathrm{l}.$$

Example 8: A thermally isolated system consists of two volumes $V_l = 1.0\,\mathrm{L}$ and $V_r = 2.0\,\mathrm{L}$ of an ideal gas separated by a movable partition. The partition is impermeable to gas, but can conduct heat. The pressures on the left and right side of the partition are $P_l = 1.0\,\mathrm{atm}$ and $P_r = 2.0\,\mathrm{atm}$, respectively. The number of moles on the left side of the partition is $n_l = 1.0\,\mathrm{mol}$. The temperature is the same throughout the entire system and remains constant. **a)** Calculate the number of moles on the right side of the partition. **b)** The partition is now allowed to move without the gases mixing. After equilibrium is established, the pressure on the left and right side is the same. Because the system is thermally insulated, the temperature does not change. Calculate the volumes of the left and right sides of the partition when equilibrium is achieved. What is the equilibrium pressure?

Solution: Use the ideal gas law: $PV = nRT$ so that $PV/n = RT$. Because the temperature is the same on both sides:

$$\frac{P_l V_l}{n_l} = \frac{P_r V_r}{n_r}, \quad \longrightarrow \quad n_r = \frac{P_r V_r}{P_l V_l} n_l = \frac{2.0\,\mathrm{atm} \cdot 2.0\,\mathrm{L}}{1.0\,\mathrm{atm} \cdot 1.0\,\mathrm{L}}\, 1.0\,\mathrm{mol} = 4.0\,\mathrm{mol}.$$

From the ideal gas law, $PV = nRT$, we have $P/RT = n/V$ so that

$$\frac{P_{\mathrm{eq}}}{RT} = \frac{n_l}{V_{l,\mathrm{eq}}} = \frac{n_r}{V_{r,\mathrm{eq}}}, \quad \longrightarrow \quad \frac{V_{r,\mathrm{eq}}}{V_{l,\mathrm{eq}}} = \frac{n_r}{n_l} = \frac{4.0\,\mathrm{mol}}{1.0\,\mathrm{mol}} = 4,$$

where $V_{l,\mathrm{eq}}$ and $V_{r,\mathrm{eq}}$ are the respective volumes in equilibrium. Since $V_{l,\mathrm{eq}} + V_{r,\mathrm{eq}} = 3.0\,\mathrm{L}$,

$$V_{l,\mathrm{eq}} + 4V_{l,\mathrm{eq}} = 5V_{l,\mathrm{eq}} = 3.0\,\mathrm{L}, \quad \longrightarrow \quad V_{l,\mathrm{eq}} = \frac{3.0\,\mathrm{L}}{5} = 0.6\,\mathrm{L},$$

so that $V_{r,\mathrm{eq}} = 4 \cdot 0.6\,\mathrm{L} = 2.4\,\mathrm{L}$. Use the ideal gas law for the left side of the partition,

$$P_l V_l = P_{\mathrm{eq}} V_{l,\mathrm{eq}}, \quad \longrightarrow \quad P_{\mathrm{eq}} = \frac{V_l}{V_{l,\mathrm{eq}}} P_l = \frac{1.0\,\mathrm{L}}{0.6\,\mathrm{L}}\, 1.0\,\mathrm{atm} = 1.67\,\mathrm{atm}.$$

Discussion: In this example, the temperature is constant because no energy enters or leaves the system as whole form the outside.

The energy of an ideal gas is the sum of the kinetic energies of all molecules: $U = N(3kT/2)$, or

$$U = \frac{3}{2}nRT \qquad \text{(monatomic gas)} \tag{37}$$

and

$$U = \frac{5}{2}nRT, \qquad \text{(diatomic gas)}, \tag{38}$$

for a diatomic gas, such as air.

Example 9: Calculate the internal energy of air inside a 'typical' room.

Solution: We assume for the volume $V = 6\,\mathrm{m} \times 4\,\mathrm{m} \times 2.5\,\mathrm{m} = 60\,\mathrm{m}^3$, and calculate the number of moles: $n = (750\,\mathrm{m}^3)/(22.4 \times 10^{-3}\,\mathrm{m}^3) = 2.7\,\mathrm{kmol}$. The internal energy follows

$$U = \frac{5}{2}\,2.7\,\mathrm{kmol} \cdot 8.3\,\frac{\mathrm{J}}{\mathrm{mol \cdot K}} \cdot 273\,\mathrm{K} = 15\,\mathrm{MJ}.$$

Discussion: That's about 3,600 food calories. This calculation shows that the energy content of macroscopic thermal systems is enormous.

12.4 Thermodynamics

The internal energy of a thermal system can be changed by adding ($Q > 0$), or removing heat ($Q < 0$) so that $\Delta U = Q$ and by doing work on the gas, e.g., by pumping the tire, $W = -P\Delta V$ so that $\Delta U = -\Delta W$. We find

$$\Delta U = U_f - U_i = Q - W. \tag{39}$$

This equation is the basis of the conversion of work into heat, e.g., by rubbing hands against each other. There is no macroscopic change of the hands so that $\Delta U = 0$. Since $W < 0$ [work is done on the system] and thus $\Delta Q = \Delta W < 0$, and heat is given off.

The development of thermodynamics is based on the quest to reverse the process and convert heat into work [that is, "useable" energy, e.g., to run an engine]. In an internal combustion engine in a car, the input of heat occurs during a controlled explosion of the fuel-air mixture. While work and heat are both energies that can be added and removed to a thermal system, they have entirely different qualities: work is associated with a reversible macroscopic change of the volume ΔV and heat is associated with irregular [random] motion of molecules. If the only change of an isolated system would be the conversion of heat into work, it would imply that disorder is destroyed [and order is created]. However, the time-development of the 'universe' is in one direction only ["arrow of time"] and disorder is created and order is destroyed.

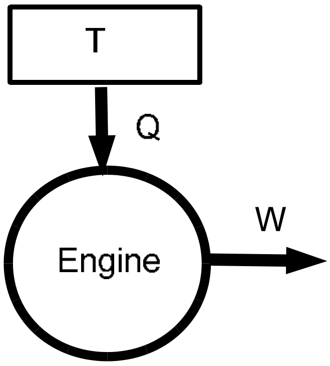
The amount of disorder in a system is quantified by the entropy S and is generally very difficult to calculate. Fortunately, only the change in entropy is important for most application and is determined by the heat added or removed,

$$\Delta S \geq \frac{Q}{T}. \tag{40}$$

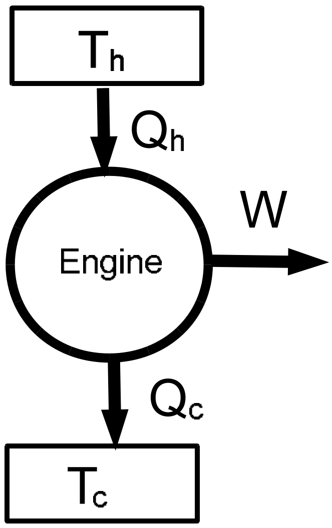
The unit of entropy is $[S] = \mathrm{J/K}$. The second law of thermodynamics then states that the entropy of the universe always increases:

$$\Delta S \geq 0. \tag{41}$$

The case $\Delta S = 0$ corresponds to reversible changes.



The engine returns to the same 'state' after one cycle [represented by the circle]. Heat is transferred at the temperature T and the output is in the form of work W. This engine would give a net loss of entropy $\Delta S = -Q/T$, and thus would violate the second law of thermodynamics. We conclude that not all heat converted into work, and that the engine must expel heat as exhaust.



We conclude that a heat engine must operate between two heat baths at temperatures T_h and T_c, respectively. Heat is removed from the bath at the higher temperature T_h and heat is expelled at the lower temperature T_c. Conservation of energy gives $Q_h = W + Q_c$ so that

$$W = Q_h - Q_c.$$

The overall change entropy is given by

$$\Delta S = -\frac{Q_h}{T_h} + \frac{Q_c}{T_c} \geq 0, \quad \longrightarrow \quad \frac{Q_c}{Q_h} \geq \frac{T_c}{T_h}.$$

Efficiency is defined as the ratio of the work [or *benefit*] divided by the input of heat [or *cost*],

$$\epsilon = \frac{W}{Q_h} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h} < 1 - \frac{T_c}{T_h}. \tag{42}$$

The maximum possible value is called the Carnot-efficiency.

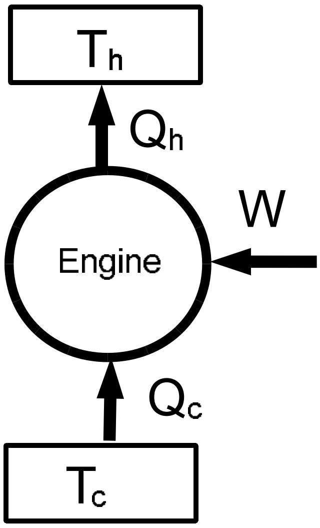
Example 10: Calculate the maximum possible efficiency for an engine that runs between boiling temperature and freezing temperature.

Solution: We have for $T_c = 273\,\mathrm{K}$ and $T_h = 373\,\mathrm{K}$. The Carnot efficieny follows

$$\epsilon = 1 - \frac{273\,\mathrm{K}}{373\,\mathrm{K}} = 0.27,$$

that is, a quarter of absorbed heat can be converted into work.

Discussion: The Carnot efficiency is an idealized case of a reversible engine. That is, an engine that runs infinitely slowly and thus 'delivers' zero power. If the power of the engine is maximized, the efficient is less than the Carnot value. One finds $\epsilon_{\rm p} = 1 - \sqrt{T_c/T_h} = 1 - \sqrt{273\,\mathrm{K}/373\,\mathrm{K}} \simeq 0.15$. This example shows that the efficiency of a 'practical' engine is much less than the corresponding Carnot efficiency.



A heat pump can be used for heating. The engine requires the input of work [e.g., *by plugging it in an electric outlet*] and removes heat Q_c from cold reservoir [outside] and delivers heat Q_h into hot reservoir [inside the house]: $Q_c + W = Q_h$. Coefficient of performance is defined as the ratio of the gain

$$\mathrm{COP} = \frac{Q_h}{W} = \frac{Q_h}{Q_h - Q_c} = \frac{1}{1 - T_c/T_h} > 1.$$

Example 11: Calculate the COP of a heater that runs between the outside at $0°\mathrm{F}$ and room temperature $60°\mathrm{F}$.

Solution: We have $T_c = 256\,\mathrm{K}$ and $T_h = 293\,\mathrm{K}$ so that

$$\mathrm{COP} = \frac{1}{1 - 256\,\mathrm{K}/293\,\mathrm{K}} \simeq 8.$$

Discussion: Note that a value grater than unity $\mathrm{COP} > 1$ does not violate conservation of energy. For one joule of work [*paid to the electric utility company!*] eight joules of heat are delivered to your living room. The difference of seven joules is provided by the cold ambient air. The heat Q_c is free. If instead you use a "space heater," one joule of work is converted into one joule of heat so that $\mathrm{COP} = 1$.

A *refrigerator* is essentially the same as a space heater, but the use is different. The cold heat bath at temperature T_c is the inside of the 'fridge' and the heat bath at the higher temperature T_h is the air of the kitchen. The fridge runs by logging it into an electric outlet. The gain is the heat Q_c removed from the inside of the fridge and the cost is the input of work:

$$\mathrm{COP} = \frac{Q_c}{W} = \frac{Q_c}{Q_h - Q_c} = \frac{1}{T_h/T_c - 1}.$$

Example 12: The inside of a refrigerator is $5°\mathrm{C}$ and the ambient air of the kitchen is $25°\mathrm{C}$. Find the COP of the refrigerator.

Solution: We have $T_c = 278\,\mathrm{K}$ and $T_h = 298\,\mathrm{K}$. Then

$$\mathrm{COP} = \frac{1}{298\,\mathrm{K}/278\,\mathrm{K} - 1} \simeq 14.$$

That is, for one joule of work [paid to the electric utility company], 14 joules of heat are removed from the inside the fridge, and 15 joules of heat are delivered to the ambient air.